OPERATIONS MANAGEMENT AND MARKETING INTERFACE: MAKING SUPPLY CHAIN DECISIONS UNDER VARIOUS MARKETING STRATEGIES

By

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To my parents, Dr. Magdi and Roblyn Hanna, for their continuous support and encouragement to aim high and work hard, and to

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This dissertation focuses on supply chain management (SCM) decisions under various marketing strategies in the Operations Management (OM) / Marketing interface research area. It is composed of three primary research chapters. The first research chapter examines optimal inventory and pricing decisions under advance selling. Advance selling is a marketing strategy in which consumers have a chance to reserve a product in an advance sales period which occurs prior to the sales period. The retailer in this scenario must make an inventory order decision before the advance sales period begins to best meet demand in both the advance sales and consumption periods. I derive optimal inventory and pricing policies. The second research chapter focuses on optimal pricing and time-to-market decisions in a new product technology (NPD) environment. I consider two generations of a new technology product considering both price and diffusion effects on sales. I derive optimal pricing and time-to-market decisions for three different sales functions. The final research chapter considers the innovation speed of new technologies in a pricing and time-to-market model. I determine the optimal number of generations to offer of a new product in this scenario. All three research chapters contribute to the OM/Marketing research literature by solving business problems from a combined OM and Marketing perspective.

CHAPTER 1
INTRODUCTION

In this chapter, I define the general area of research for the dissertation and introduce the various research chapters.

1.1 Operations Management (OM) / Marketing Interface Research

The focus of this dissertation work is on making joint operations and marketing decisions for standard supply chain problems in various marketing settings. The importance of OM/Marketing interface work has been highlighted in recent literature in both the OM and Marketing fields. This new academic perspective has been motivated by problems arising in industry when OM and Marketing departments discover their separate goals may be conflicting and have negative effects on the company as a whole. Thus OM/Marketing interface research strives to simultaneously consider OM and Marketing decisions and objectives in order to maximize benefits for the entire firm.

1.2 Inventory Management under Advance Selling: Optimal Order and Pricing Policies

Advance selling is a marketing strategy in which an advance sales period precedes the spot period, and customers are uncertain about their future product valuation. I consider an inventory management decision, in which I decide an inventory order quantity as well as a portion of this inventory to reserve for advance sales. I use an expected profit maximization model to find the optimal order and pricing policies.

I find the optimal advance sales inventory level to be an extreme point solution. This leads to a "go/no-go" advance sales decision in which I either spot sell (sell in the spot period) to all customers and advance sell to no one or advance sell (sell in the advance sales period) to almost all customers. I derive several analytical results and perform numerical experiments which examine the behavior of the optimal order policy and provide sensitivity analysis on the model parameters.

1.3 Multi-Generation Pricing and Timing Decisions in New Product Development

When planning for the introduction of a stream of new products into the marketplace, managers must consider both the timing and dynamic pricing decisions to determine an appropriate entry strategy into the marketplace. Literature in the new product development (NPD) area has addressed optimal timing of multiple generations of products and the dynamic pricing decisions independently. However, no analytic results have been developed when these decisions are considered simultaneously.

In Chapter 4, I develop an analytical model of coordinated product introduction and pricing decisions when there are two generations of a new product under consideration. Factors driving the decisions include the unit sales and cost relationships for each generation as well as NPD costs for introducing the next generation of products. I derive analytic results that characterize the optimal timing and pricing strategies. In addition, I identify an optimal threshold value for the length of the planning horizon which dictates the new product introduction strategy. Further insights are obtained for a special case of the model where the two generations of products have similar sales and pricing characteristics.

1.4 Optimal Number of Generations for a Multi-Generation Pricing and Timing Model in New Product Development

In Chapter 5, I seek to find the optimal number of generations to introduce under the pricing and time-to-market NPD model developed in Chapter 4. In Chapter 5, I consider a multi-generation new technology product with sales as a function of both price and diffusion. In addition to solving for the optimal pricing policy and optimal time to market for each generation, I find the optimal number of generations that a firm should introduce. The new product development (NPD) literature has addressed the time-to-market decision for multi-generation products and simultaneous optimal pricing policies have recently been introduced to the Operations Management (OM) / Marketing Interface literature. However, very little work has been done on the optimal number of generations to introduce. In comparison with innovation speed (or clockspeed) papers, I consider a specific additive model of sales with both pricing and diffusion effects and simultaneously solve for optimal pricing, timing, and number of generations for a maximum profit objective. The analytics employ optimal control theory. An extensive numerical analysis is also performed.

1.5 Overview of the Dissertation

The remainder of the dissertation is organized as follows. In Chapter 2, I present a detailed review of the literature. I discuss papers which pertain to the importance and scope of OM/Marketing Interface work, as well as papers related to the research done in each of the dissertation chapters. The next three chapters comprise the main research of the dissertation. In Chapter 3, I discuss the work titled "Inventory Management Under Advance Selling: Optimal Order and Pricing Policies", in Chapter 4, I present "Multi-Generation Pricing and Timing Decisions in New Product Development", and in Chapter 5, I present the research for "Optimal Number of Generations for a Multi-Generation Pricing and Timing Model in New Product Development". In each of these research chapters, I introduce the problem, describe the model and assumptions, perform the analysis, and review numerical experiments. The final chapter of the dissertation, Chapter 6, provides the conclusions for each of these research chapters and discusses future research extensions for these works as well as in the general OM/Marketing Interface research area.

CHAPTER 2
LITERATURE REVIEW

In this chapter, I present a review of the literature related to the research in each of the main chapters. I also discuss the literature pertaining to the general OM/Marketing Interface research area.

2.1 OM/Marketing Interface

Several papers in the recent literature of both the OM and Marketing fields have highlighted the importance of interface research. Some survey papers such as Karmarkar (1996) [1] and Balasubramanian, et.al. (2004) [2] as well as some empirical work such as Hausman, et.al. (2002) [3] examine problems caused by miscommunication between business departments and disjoint objectives in the supply chain. They show how a marketing push for higher prices and quicker turnover can have a negative influence on operations cost and production requirements. Likewise operations priorities such as lower costs and other inventory and production motivations can hurt marketing goals. The academic community has now been challenged by industry to help coordinate these multi-disciplinary goals by researching joint decision problems. In the extensive work of Eliashberg, et.al. (1993) [4], these new marketing-production joint decision problems are discussed as this new optimization area continues to expand. Below, I refer to more specific works corresponding to each of the research chapters in this dissertation.

2.2 Inventory Management under Advance Selling: Optimal Order and Pricing Policies

The advance sales strategy is most clearly defined in the marketing literature. There is however also related literature in the revenue management and inventory research areas. In this section , I discuss the relative literature for advance selling and operations management decision making.

Xie and Shugan (2001) [5] write a comprehensive paper describing the idea and benefits behind advance selling. They show that advanced selling is profitable in the general scenario of unknown future consumer valuation. Their paper assumes that the consumer's valuation function is Bernoulli with high, H, or low, L, values occurring with probability α and $(1-\alpha)$, respectively. They also assume that a deterministic N customers arrive in both the advance sales and spot sales period. They state that if the marginal cost of offering advanced selling is low to medium, then advanced selling should be offered and the advance sales price should be set to the expected price $\alpha H + (1 - \alpha)L$ and the spot price should be set to H. They then describe optimal advance selling strategies for various capacity scenarios. They show that if capacity is limited, but large, then the advance sales price can be set to a premium and the spot price should be set to L. In another limited capacity scenario with medium capacity, the advance sales price should be set to the expected price $\alpha H + (1 - \alpha)L$ and the spot price should be set to H. In the third limited capacity scenario with small capacity, there should be limited advanced sales with the advance sales price set to the expected price $\alpha H + (1 - \alpha)L$ and the spot price set to H. The modeling techniques used in their paper include dynamic programming to find the optimal spot price, followed by the optimal advance sales price. I advance their modeling assumptions by considering a more realistic description of customer arrivals and dynamic customer valuations.

Shugan and Xie (2004) [6] write another paper describing the advance selling in service industries. This paper is quite similar to Shugan and Xie (2001), but it uses an exponential customer valuation function instead of the Bernoulli distribution. It also gives a comparison of advanced sales to yield management systems (YMS). It explains that YMS are limited because it requires binding capacity constraints, very low marginal cost of additional sales, and an inverse relationship between consumer price sensitivity and customer arrival time. Their paper also reviews various scenarios in which advanced sales is beneficial, again including limited capacity (limited advanced sales and premium advanced sales). Their modeling in this paper uses a buyer decision tree when advanced sales are priced at a premium and a simple 2-case model is used to compare advanced sale and spot sale profits.

As discussed in the Shugan and Xie (2004) paper, advance selling is similar to yield management, or revenue management. Thus, there are some related ideas found in the revenue management literature. The Desiraju and Shugan (1999) paper [7] clearly defines yield management systems (YMS). Pricing strategies based on YMS may indeed be profitable. They consider discounting, overbooking (which is shown to be beneficial in some cases), and limited early sales for capacity-constrained services. In YMS, there are two distinct market segments: price-insensitive, which have high valuations, and price-sensitive, which have low valuations. There are also three distinct service classes. Class A represents early arrivals from the price-sensitive market, Class B represents early arrivals from the price-insensitive market, and Class C represents early arrivals from both markets (that is, there is no distinction among the market segments). Their results show that for Class A, limited early sales at lower/increasing prices is the best strategy, for Class B, unlimited early sales at higher/decreasing prices is best, and for Class C, YMS is not profitable so the firm should sell at their best price. Their modeling techniques include dynamic programming: for a given optimal spot price, they consider total profits to find the best advance sales price. They leave some open extensions to their work, such as considering competition, additional market segments, channels of distribution, different quality levels, and signaling on YMS.

This paper makes the clear distinction between revenue management and advance selling in the assumption requirements. Advance selling only requires that customers are uncertain about their future valuation, whereas revenue management assumes a certain timing of different customer segments and usually a capacity constraint. Revenue management seeks more to allocate a given capacity among customer segments by creating a price menu. In advance selling, customers are homogeneous and may arrive at any time to the market.

Ideas similar to advance selling can also be found in the inventory literature.

Tang, et. al. (2004) [8] describe an inventory strategy called Advance Booking Discounts (ABD). They discuss the benefits of this strategy over Quick Response (QR), a similar inventory strategy. In QR, a firm produces in a first period (based on retailer estimates) and also produces in a second period (based on retailer updated estimate after some early sales). In ABD, however, a firm produces in only one period (after pre-season discount sales). Their results show that profits from the ABD strategy with forecast updating are greater than profits of ABD without forecast updating. They also show that profits of ABD can be greater than a base case scenario (where no ABD strategy is employed). They find that discount prices with forecast updating are greater than the discount prices without updating. Their model seeks to maximize profit. They achieve newsvendor results. Some extensions to their work include finding the optimal length of the ABD advanced period, considering ABD premiums instead of discounts, and considering capacity constraints.

In an earlier paper, Iyer and Bergen (1997) [9] introduce the benefits of QR. They discuss conditions under which it is Pareto-improving (both retailer and manufacturer benefit or are as well off). With QR, there is an initial demand observation period, then the retailer places order, then the lead time for production/shipping occurs, then the order is received and the season begins. Their results show that QR is always beneficial for the retailer, but not necessarily for the manufacturer. In order to make QR Pareto-improving, the firm must examine their service level, price, and volume commitments. QR also depends on the accuracy of the demand estimate from the initial observation period. They use a profit-maximization model. They find a newsvendor solution for an optimal inventory level and service level. They determine that in order to find expected profits for QR, the firm should choose an order size which maximizes profit for a posterior demand distribution. Some extensions to their work include competition, multiple sales periods, and multiple manufacturers. They could also consider if QR is beneficial in other markets?

Also, what would be the effect of the demand observation period length on the QR benefit to the retailer and manufacturer?

In general, there seems to be an open research direction for applying operations management decisions to sales strategies such as advance selling. My research contributes to the literature by presenting a more realistic model of consumer's valuation with uncertainty, finding an optimal pricing strategy using dynamic pricing for the advance sales period.

2.3 Multi-Generation Pricing and Timing Decisions in New Product Development

The literature related to the model in Chapter 4 can be divided into the following categories: Sales Behavior for Single and Multiple Generations, Pricing Decisions, and Timing Decisions research. While most of this literature addresses a subset of these topics in isolation, I combine the dynamic pricing and generational timing decisions into a single model.

A large body of literature addresses the sales and/or diffusion process for products in both single and multiple generation settings. Many of these stem from Bass (1969) [10] who describes the diffusion process for a single generation of products as a function of both innovation (i.e. early adopters) and imitation (i.e. later buyers). This empirically based model has been shown to be a robust characterization for the diffusion process of durable goods including growth, maturity and decline phases of the product life cycle. Norton and Bass (1987) [11] create a multiple generation version of the original Bass model which incorporates substitution effects and increasing sales across the generations. One key facet of their model utilized here concerns the assumption that the innovation and imitation parameters remain the same across multiple generations of the same product.

Another body of literature addresses the dynamic pricing problem associated with the introduction of a single generation of a new product into the marketplace. Kalish (1983)

[12] analyzes different scenarios illustrating the impact of discounting, learning effects and diffusion on the dynamic pricing problems. One of the scenarios that Kalish examines is the case of a durable good where positive word of mouth stimulates demand early in the life cycle, while saturation takes over and demand increases later in the life cycle. The optimal dynamic price under these conditions will start relatively low, increase as long as the word-of-mouth effect overcomes the saturation effect, and then decrease for the remainder of the planning horizon. Bass et al. (1994) [13] propose a generalized version of the original Bass model which includes the dynamic effects of pricing and/or advertising on product diffusion. Krishnan et al. (1999) [14] uses the General Bass Model (GBM) to identify an optimal price path for a new generation of products. In contrast to previous literature, these authors find that the optimal dynamic price does not follow a traditional sales growth pattern, but (in many cases) is decreasing. Sethi and Bass (2003) [15] also find that both price and sales rate decline over time for a special case of GBM. Teng and Thompson (1996) [16] consider the impact of both quality and price simultaneously on cumulative sales and profit. Both Bass et al. (1994) [13] and Krishnan et al. (1999) [14] offer comprehensive overviews of the literature which incorporates price and/or advertising factors into diffusion models.

The dynamic pricing problem has been extended to address optimal pricing strategies for multiple generations of new products. However, the entry time for the new generations is considered exogenous or given in these models. Padmanabhan and Bass (1993) [17] analyze a model which captures substitution and cannibalization effects for a firm that introduces two generations of new products during a finite planning horizon. Results from this model show that the actual prices changed at each instant of time for the two products are significantly different with the consideration of product line issues in the profit maximization problem as compared to the single generation models such as Kalish (1983) [12]. In their conclusion section, these authors also comment that, “The demand model used in (this) analysis assumes that the time of entry of the second product is determined exogenously.. Endogenous consideration of this issue would be a very worthwhile contribution to the literature." Kornish (2001) [18] also considers the pricing problem for two generations of a product based on the consumers valuations for each generation.

Several authors address the problem of optimal introductory timing for multiple generations of new products into a marketplace. Based on numerical analysis, Mahajan and Muller (1996) [19] develop a now or at maturity rule for the introduction timing of a second generation of products. Specifically, they find that the firm should either introduce the second generation as soon as it is available or delay its introduction to the maturity stage of the preceding generation. Another model which determines the optimal timing of the introduction of a second product into the marketplace is developed in Carrillo and Franza (2004) [20], who also consider the impact of both process development and product development activities on this decision. Morgan et al. (2001) [21] consider a quality versus time-to-market tradeoff when multiple generations are introduced. Carrillo (2004) [22] and Carrillo (2005) [23] address the optimal number of generations to introduce during a given planning horizon and analyze the impact of dynamic profit margins on the timing decision. However, none of these models considers the impact of timing simultaneously with pricing as decision variables.

2.4 Optimal Number of Generations for a Multi-Generation Pricing and Timing Model in New Product Development

The importance of considering clockspeed as a component of NPD decision making has been highlighted in the recent marketing literature. In Carbonell and Rodriguez (2006) [24], the authors analyze the effect that innovation speed has on the perception of marketing advantage. They recognize that innovation speed is an equivalent, if not more important, marketing characteristic affecting NPD sales. Other marketing literature, such as Nadler and Tushman(1999) [25], Pearce (2002) [26], and Lambert and Slater (1999) [27] have discussed the direct impact that innovation speed, or clockspeed, may have on the sales rate. In fact, some empirical work has shown that sales may increase as the number of generations increases.

In the operations management (OM) literature, limited research has been done on the optimal number of generations to introduce in an NPD scenario. In Carrillo (2005) [23], the author solves for the optimal clockspeed under various sales curves. Conditions are derived to determine when firms may have an incentive to increase their clockspeed. In another paper, by Souza, Bayus, and Wagner (2004) [28], innovation speed is compared with quality decisions and time to market.

Comparative models can be found in Lukas and Menon (2004) [29], who look at the joint quality and innovation speed problem, and Dahan and Mendelson (2001) [30] who examine concept testing in NPD. Another OM work by Xu and Li (2007) [31] address the joint technology investment and innovation decision in assemble-to-order systems.

In Chapter 4, optimal pricing and time-to-market decisions are derived for a two-generation new technology product. This chapter makes a substantial contribution to the OM/Marketing Interface literature for the NPD marketing scenario. Chapter 5 also makes a significant contribution to the OM/Marketing Interface literature by solving both pricing and time-to-market decisions in addition to solving for the optimal number of generations of an NPD product.

CHAPTER 3
INVENTORY MANAGEMENT UNDER ADVANCE SELLING: OPTIMAL ORDER AND PRICING POLICIES

3.1 Introduction

Advance selling is a marketing strategy in which an advance sales period precedes the standard consumption, or spot, period. The advance sales period is used to increase sales by offering customers a chance to commit early to purchasing at what is usually a discounted price. This strategy takes advantage of customers being uncertain about their future product valuation. Advance selling has become increasingly more popular with recent technologies such as smart cards and online booking (see the Economist (2005) [32]).

Most applications of advance selling are in the service industry. For example, consider ticket sales for a concert. Ticket prices may be $50 at the door but on sale for $30 if bought in advance. Other examples of advance selling in the service industry may include conference registration, movie tickets, and vacation packages.

Much of the literature in advance selling has to do with finding the optimal pricing policy for the advance and spot sales periods. I am interested in extending this marketing analysis to include operations decisions, specifically an inventory management decision.

I assume that a one-time inventory order must be placed at the beginning of the advance sales period. The inventory will not arrive until consumption, which occurs at the end of the spot period. I may consider a situation in which there is a long lead time for an inventory order and no opportunity to place another order before consumption.

An example of this scenario may occur in the toy industry. I may need to decide on an order quantity of toys for an upcoming sales season. Demand may be uncertain in that customers have not yet realized their future valuation of a particular toy. If the toys are produced in a distant facility with a long lead time, I may only have one order opportunity. An advance selling strategy would be to offer reservations of some portion of these toys and to reserve the remaining portion for in-store sales.

Another example may occur in event planning. I may need to reserve a location for an upcoming event with uncertain demand. Again, customers may be uncertain of their future valuation of attending this event. The order quantity in this case would be equivalent to the capacity of the location I reserve. Since many event locations must be reserved prior to the event with some kind of non-refundable deposit, I may consider this to be a one-time order decision. Advance selling in this scenario would include reserving some of this capacity for advance sales tickets and leaving the remaining for at-the-door sales.

A third example may be in real-estate marketing of new condominium development. When developing a new condominium complex, I must decide ahead of time how many units to build. Since I cannot add or subtract units once construction begins, I may consider this to be a one time order decision. Demand is uncertain in that customers are unsure of their future valuation of purchasing a condominium unit. The advance selling strategy would involve reserving some units to sell in advance and keeping the remaining portion available for sale after the condominiums have been completed.

In addition to deciding the inventory order quantity, I also consider what portion of this inventory to reserve for advance sales. I use analytical and numerical results to better understand when it is optimal for the firm to offer advance sales, and if so, how many advance sales to offer. I also seek to determine the optimal pricing policy for the advance and spot sales periods. I use an expected profit maximization model to find the optimal order and pricing policies.

The rest of the chapter is organized as follows. In Section 2, I describe the model and assumptions. In Section 3, I perform the analysis and give structural solutions to the optimal inventory and pricing policies. In Section 4, I describe several numerical experiments and discuss sensitivity analysis. In Section 5, I consider an extension for a different customer valuation distribution and describe related numerical experiments.

Please refer to the Chapter 2 for a review of the related literature.

3.2 Model

There are two periods in which a firm sells a product or service: the advance sales period and the spot period. The advance sales period precedes the spot period, and consumption occurs at the end of the spot period. I have a market of size M, with units equal to the number of customers. A portion of this market will arrive to the advance sales period, N_a, and the remainder will arrive to the spot period, $N_s = M - N_a$. As with the market size, N_a and N_s represent a number of customers.

The customers that arrive to the advance sales period will be offered the product (or service) at a price p_a. I assume that the spot price p_s is also announced to the customers in the first period.

Customers decide whether or not to buy the product based on their valuation of the product V, measured as a dollar value. I assume that the true valuation of the product is not realized until the spot period. Thus, during the advance period, customers are uncertain about their future valuation but know the distribution of the future valuation and the expected future valuation, $E[V]$. Customers that arrive to the advance sales period must thus decide whether to buy the product in the advance period or wait until the spot period based on their expected future valuation, the advance sales price, and the announced spot price. During the spot period, customers decide whether or not to buy the product based on the spot price and their realized product valuation.

I consider an inventory management decision under this advance selling scenario. The firm must place an inventory order Q at the beginning of the advance sales period. Without loss of generality, I assume that this inventory order is delivered at consumption, which occurs at the end of the spot period. Some portion of this inventory, X_a, is reserved for the advance sales demand, and the remaining inventory, $X_s = Q - X_a$, is used to satisfy spot sales demand, where I have $X_a \leq Q \leq M$. Consider the timeline in Figure 3-1.

I seek to determine the optimal values for the order quantity Q, the advance sales inventory portion X_a, the advance sales price p_a and the spot price p_s such that total expected profit is maximized.

Notations

M	Market size
N_a	Number of customers arriving to advance sales period
N_s	Number of customers arriving to spot sales period
p_a	Advance sales price
p_s	Spot sales price
V	Customer valuation of product at consumption
Q	Inventory order quantity
X_a	Inventory allocated to advance sales
X_s	Inventory allocated to spot sales

Let us now derive the expressions for the advances sales and spot sales demand. The spot period demand D_s is a Binomial random variable with the number of events equal to the number of spot period customers $N_s = M - N_a$ and the probability of success, or probability of purchase, dependent on the customer valuation, $Pr\{p_s \leq V\}$.

$$D_s \sim Binomial(N_s, Pr\{p_s \leq V\}) \tag{3–1}$$

In the advance sales period, a customer will only buy the product if the expected utility of an advance purchase is greater than or equal to the expected utility of waiting to

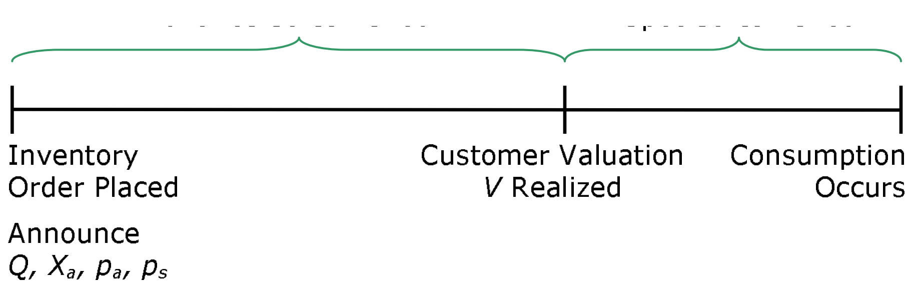

Figure 3-1. Timeline

purchase in the spot period. The utility of an advance purchase is the difference between a customer's valuation and the advance sales price.

$$U_a = V - p_a \tag{3–2}$$

Thus I have the expected utility of an advance purchase as follows.

$$E[U_a] = E[V] - p_a \tag{3–3}$$

I assume that customers are aware that there is limited inventory available and thus there is a risk that there may not be enough inventory to satisfy all demand. This information will affect the customer's utility of waiting to purchase in the spot period. Let β represent the probability that a customer will find available inventory in the spot period. I can think of this probability as the firm's demand fill rate. I define β as the ratio of satisfied demand to total demand. I define satisfied demand as the minimum of the available inventory and the demand: $\min(Q - X_a, D_s)$, where D_s is the demand in the spot period.

$$\beta = \frac{\min(Q - X_a, D_s)}{D_s} \tag{3–4}$$

The utility of waiting to purchase in the spot period is then the positive difference between their valuation and the spot price, $(V - p_s)^+$ if inventory remains, or 0 if there is no inventory. Using β as defined above as the probability of having available inventory in the spot period, I define the utility of waiting to purchase in the spot period as follows.

$$U_s = \begin{cases} (V - p_s)^+, & \text{with probability } \beta, \\ 0, & \text{with probability } 1 - \beta. \end{cases} \tag{3–5}$$

Although β is a function of the spot demand, D_s, which is dependent on the customer valuation V, I assume the valuations which determine the spot demand are for customers who will sport purchase, which does not include the advance sales customer I am currently considering. That is, the valuations considered in the spot demand through the Binomial probability of success $Pr\{p_s \leq V\}$ are independent of the valuation of the decision-making advance sales customer whose valuation appears in $(V - p_s)^+$. Thus, I can calculate the expected utility of waiting to purchase in the spot period as follows.

$$E[U_s] = E[(V - p_s)^+]E[\beta] \tag{3–6}$$

Thus, a customer will decide to purchase in the advance sales period if and only if $E[U_a] \geq E[U_s]$. Evaluating this comparison translates this condition into a maximum advance sales price, $\hat{p}_a$ for which a customer will decide to advance purchase.

$$E[U_a] \geq E[U_s] \tag{3–7}$$

$$E[V] - p_a \geq E[(V - p_s)^+]E[\beta] \tag{3–8}$$

$$p_a \leq E[V] - E[(V - p_s)^+]E[\beta] = \hat{p}_a \tag{3–9}$$

Thus, if I set the advance sales price p_a equal to this maximum advance-purchase-inducing price $\hat{p}_a$, then all customers who arrive to the advance sales period will choose to advance purchase. Since I will decide the portion of the inventory reserved for advance sales, X_a, I can assume that all of this inventory will be sold in the advance period when $p_a \leq \hat{p}_a$. Thus, I can consider the number of advance sales customers to be equivalent to this advance sales inventory portion: $N_a = X_a$, and the number of customers who decide to wait is zero. That is, I decide X_a and then advance sell to that many customers, $N_a = X_a$, knowing that they will all agree to advance purchase if $p_a \leq \hat{p}_a$.

The advance sales demand D_a is thus equal to the number of customers who arrive to the advance sales period ($D_a = N_a$), which is the same as the portion of inventory that I reserve for advance sales ($N_a = X_a$).

I can now write my profit-maximization objective function. We assume a unit advance sales cost c_a, unit order cost c, and no salvage value. The profit expression considers profit earned from the advance sales purchases X_a, revenue from spot sales, and the inventory order cost.

$$\Pi = (p_a - c_a)X_a + p_s \min(Q - X_a, D_s) - cQ \tag{3–10}$$

$$E[\Pi] = (\hat{p}_a - c_a)X_a + p_s(\mu_{D_s} - \Lambda_{D_s}(Q - X_a)) - cQ \tag{3–11}$$

Where Λ_{D_s} is the loss function $\int_{Q-X_a}^{\infty}(t-Q+X_a)f_{D_s}(t)dt$ and $f_{D_s}(t)$ is the Normal *pdf*. Thus, my optimization problem is to determine the optimal values for the order quantity Q, the advance sales inventory portion X_a, the advance sales price p_a and the spot price p_s such that total expected profit is maximized.

MAX

$$E[\Pi] = (\hat{p}_a - c_a)X_a + p_s(\mu_{D_s} - \Lambda_{D_s}(Q - X_a)) - cQ \tag{3–12}$$

subject to

$$0 \leq X_a \leq Q \leq M \tag{3–13}$$

Let us assume that a customer's product valuation is a Bernoulli random variable which can have a high value H with probability α or a low value L with probability $1 - \alpha$, as was done similarly in Xie and Shugan (2001) [5]. A customer will only buy the product if the price is less than or equal to this valuation. In the spot period, this valuation is realized, thus a customer will buy the product with probability $Pr\{p_s \leq V\}$. With the Bernoulli definition of valuation, this probability is:

$$Pr\{\text{spot purchase}\} = Pr\{p_s \leq V\} = \begin{cases} 0, & \text{if } p_s > H; \\ \alpha, & \text{if } p_s = H; \\ \alpha, & \text{if } L < p_s < H; \\ 1, & \text{if } p_s \leq L. \end{cases} \tag{3–14}$$

It is clear that it will never be profitable to offer a spot price $p_s > H$, since no customers would purchase. Since the probability of a spot purchase is the same for $p_s = H$

and $L < p_s < H$, it would be more profitable to offer a spot price of $p_s = H$. Offering a spot price of $p_s = L$ would be more profitable than any price lower than L. Thus, from this point forward I will assume that the firm considers offering a spot price p_s of either H or L, but not any value in between. I will therefore perform my analysis for these two cases.

I can now calculate the spot demand, D_s, to be as follows.

$$D_s \quad \begin{cases} \sim Binomial(N_s, \alpha), & \text{if } p_s = H; \\ = N_s, & \text{if } p_s = L. \end{cases} \tag{3–15}$$

$$E[D_s] = \begin{cases} N_s\alpha = (M - X_a)\alpha, & \text{if } p_s = H; \\ N_s = M - X_a, & \text{if } p_s = L. \end{cases} \tag{3–16}$$

For the case when $p_s = H$, I will approximate the Binomial with a Normal distribution with mean μ_{D_s} and σ_{D_s} defined as follows.

$$D_s \sim Normal(\mu_{D_s}, \sigma_{D_s})$$

$$\mu_{D_s} = (M - X_a)\alpha \tag{3–17}$$

$$\sigma_{D_s} = \sqrt{(M - X_a)\alpha(1 - \alpha)} \tag{3–18}$$

$E[\beta]$ can now be calculated as follows:

$$E[\beta] = \frac{E[\min(Q - X_a, D_s)]}{E[D_s]} \tag{3–19}$$

$$= \begin{cases} \frac{E[D_s - (D_s - Q + X_a)^+]}{E[D_s]}, & \text{if } p_s = H; \\ \frac{\min(Q - X_a, M - X_a)}{M - X_a}, & \text{if } p_s = L. \end{cases} \tag{3–20}$$

$$= \begin{cases} 1 - \frac{\Lambda_{D_s}(Q - X_a)}{(M - X_a)\alpha}, & \text{if } p_s = H; \\ \frac{(Q - X_a)}{M - X_a}, & \text{if } p_s = L. \end{cases} \tag{3–21}$$

Note that I evaluate $\min(Q - X_a, M - X_a) = Q - X_a$ based on the fact that $X_a \leq Q \leq M$. Note also that I use the approximation $E[\beta] = \frac{E[min(Q-X_a,D_s)]}{E[D_s]}$, although it has been shown in the literature that this will yield larger values than the expected value $E[\frac{min(Q-X_a,D_s)}{D_s}]$.

I now return to my definition of the maximum advance sales price, $\hat{p}_a = E[V] - E[(V - p_s)^+]E[\beta]$. Using the Bernoulli distribution for customer valuation, I calculate the following.

$$E[V] = H\alpha + L(1-\alpha) \quad (3\text{–}22)$$

$$E[(V-p_s)^+] = \sum_{v=p_s}^{H}(v-p_s)Pr\{V=v\} \quad (3\text{–}23)$$

$$= \begin{cases} 0, & \text{if } p_s = H; \\ (H-L)\alpha, & \text{if } p_s = L. \end{cases} \quad (3\text{–}24)$$

Now I can express $\hat{p}_a$ using the above definitions and the expressions for $E[\beta]$ as follows.

$$\hat{p}_a = \begin{cases} H\alpha + L(1-\alpha), & \text{if } p_s = H; \\ H\alpha + L(1-\alpha) - (H-L)\alpha(\frac{Q-X_a}{M-X_a}), & \text{if } p_s = L. \end{cases} \quad (3\text{–}25)$$

$$= \begin{cases} L + (H-L)\alpha, & \text{if } p_s = H; \\ L + (H-L)\alpha(\frac{M-Q}{M-X_a}), & \text{if } p_s = L. \end{cases} \quad (3\text{–}26)$$

The expected profit $E[\Pi]$ for Bernoulli customer valuations is then as follows.

$$E[\Pi] = \begin{cases} (L + (H-L)\alpha - c_a)X_a & \\ \quad + H((M-X_a)\alpha - \Lambda_{D_s}(Q-X_a)) - cQ, & \text{if } p_s = H; \\ (L + (H-L)\alpha(\frac{M-Q}{M-X_a}) - c_a)X_a & \\ \quad + L(Q-X_a) - cQ, & \text{if } p_s = L. \end{cases} \quad (3\text{–}27)$$

3.3 Analysis

3.3.1 Optimal Order Quantity $Q^*(X_a)$

First let us find the optimal order quantity Q^* as a function of X_a. I will solve for each spot price case separately ($p_s = L$ and $p_s = H$).

For the case when $p_s = L$, I find expected profit $E[\Pi(L)]$ to be linear in Q. $E[\Pi(L)]$ is increasing in Q, under the condition $\frac{-X_a(H-L)\alpha}{M-X_a} + L - c > 0$, and decreasing in Q otherwise. I have the following theorem.

Theorem 1. *For $p_s = L$ and Bernoulli customer valuations, the expected profit $E[\Pi(L)]$ is linear increasing in Q for $\frac{-X_a(H-L)\alpha}{M-X_a} + L - c > 0$. Thus the optimal order quantity Q^* for a given X_a and $p_s = L$ is:*

$$Q^*(X_a, L) = \begin{cases} M, & \text{if } \frac{(H-L)\alpha}{L-c} < \frac{M-X_a}{X_a}, \\ 0, & \text{otherwise.} \end{cases} \quad (3\text{–}28)$$

(See Appendix A for the proof.)

That is, for a low spot price, when the above condition is met, the optimal order quantity is equal to the entire market. Otherwise I do not order anything. This "all or nothing" result is due to the Bernoulli customer valuation and low spot price. When the spot price is low ($p_s = L$), the advance sales price is also low ($\hat{p}_a = L$). Thus, I order enough for the entire market since everyone will buy.

In the case when $p_s = H$, I find expected profit $E[\Pi(H)]$ to be concave in Q with the optimal inventory size $Q^*(X_a, H)$ as follows.

Theorem 2. *For $p_s = H$ and Bernoulli customer valuations, the expected profit $E[\Pi(H)]$ is concave in Q and the optimal order quantity Q^* for a given X_a and $p_s = H$ is:*

$$Q^*(X_a, H) = F_{D_s}^{-1}\left(\frac{H-c}{H}\right) + X_a \quad (3\text{–}29)$$

(See Appendix A for the proof.)

In this case, since the spot price is high, I cannot be certain how many customers will buy in the spot period. That is, I may have customers in the spot period with a realized valuation lower than H. I see that $Q^*(X_a, H)$ resembles the standard newsvendor solution, which captures the effect of this demand uncertainty, with an additional quantity for the advance sales inventory, X_a.

Thus I have found the optimal order quantity Q^* as a function of X_a to be:

$$Q^*(X_a) = \begin{cases} F_{D_s}^{-1}(\frac{H-c}{H}) + X_a, & \text{if } p_s = H; \\ M, & \text{if } p_s = L \text{ and } \frac{(H-L)\alpha}{L-c} < \frac{M-X_a}{X_a}, \\ 0, & \text{if } p_s = L, \text{ otherwise.} \end{cases} \tag{3–30}$$

3.3.2 Optimal Advance Sales Inventory $X_a^*(Q)$

Now let us find the optimal advance sales inventory quantity X_a^* as a function of Q. I will solve for each spot price case separately ($p_s = L$ and $p_s = H$).

For $p_s = L$, I find expected profit $E[\Pi(L)]$ to be convex in X_a. Since I want to maximize expected profit, the optimal advance sales inventory $X_a^*(Q, L)$ is thus an extreme point solution, with $0 \leq X_a(L) \leq Q$.

Theorem 3. *For $p_s = L$ and Bernoulli customer valuations, the expected profit $E[\Pi(L)]$ is convex in X_a. Thus the optimal advance sales inventory level X_a^* for a given Q and $p_s = L$ is an extreme point solution.*

$$X_a^*(Q, L) = \begin{cases} Q, & \textit{if } (H-L)\alpha \geq c_a, \\ 0, & \textit{otherwise.} \end{cases} \tag{3–31}$$

(See Appendix A for the proof.)

Since I have an extreme point solution, I either reserve all of my order quantity for advance sales or I do not advance sell at all. The above condition states that as long as the cost of advance selling is relatively low, I will advance sell to everyone. This seems reasonable since the spot price is low. Thus, there is no advantage to reserving inventory for the spot period if I can sell everything for the same price in the advance sales period.

For $p_s = H$, I also find the optimal advance sales inventory $X_a^*(Q, H)$ to be an extreme point solution, with $0 \leq X_a(H) \leq Q$.

Theorem 4. *For $p_s = H$ and Bernoulli customer valuations, the optimal advance sales inventory level X_a^* for a given Q and $p_s = H$ is an extreme point solution.*

$$X_a^*(Q,H) = \begin{cases} Q, & \textit{if } Q[L(1-\alpha) - c_a] \geq H[\mu_{D_s} - \Lambda_{D_s}(Q)], \\ 0, & \textit{otherwise.} \end{cases} \tag{3–32}$$

Under the following conditions.

$$c \leq 0.31H \tag{3–33}$$

$$\sigma_{D_s} \leq \frac{1}{4}k - \frac{1}{8} \tag{3–34}$$

$$\alpha_{LB} \leq \alpha \leq \alpha_{UB} \tag{3–35}$$

where

$$\alpha_{LB} = \begin{cases} \frac{1-\sqrt{1-\frac{\sigma_{D_s}}{k-1/2}}}{2} & , \textit{for } \frac{c}{H} \geq 0.00169 \\ 1 - \frac{\mu_{D_s}(2\mu_{D_s}+3k\sigma_{D_s})-2}{k^2\mu_{D_s}} & , \textit{otherwise} \end{cases} \tag{3–36}$$

$$\alpha_{UB} = \frac{1+\sqrt{1-\frac{\sigma_{D_s}}{k-1/2}}}{2} \tag{3–37}$$

(See Appendix A for the proof.)

Again, since I have an extreme point solution, I either reserve all of my order quantity for advance sales or offer no advance sales at all. The condition for advance selling all of the order quantity is dependent on how uncertain the spot demand is. That is, the lower the chance of earning sales in the spot period, the riskier it is to reserve more spot sales, despite the high spot price. Thus it is more profitable to advance sell all of my inventory.

Note that the condition for $\alpha_{LB} = \frac{1-\sqrt{1-\frac{\sigma_{D_s}}{k-1/2}}}{2}$ of $\frac{c}{H} \geq 0.00169$ implies that $\sigma_{D_s} \leq 0.61$. That is, for higher cost values, or cost values closer to H, the variance of the spot demand must be small for this lower bound on α to hold. This α_{LB} value is increasing in the ratio $\frac{\mu_{D_s}}{\sigma_{D_s}}$, creating a tighter bound. The opposite is true for the other value of $\alpha_{LB} = 1 - \frac{\mu_{D_s}(2\mu_{D_s}+3k\sigma_{D_s})-2}{k^2\mu_{D_s}}$. In this case, the cost is relatively insignificant ($\frac{c}{H} \leq 0.00169$) and the variance of the spot demand can be much higher ($\sigma_{D_s} \geq 0.61$) for the lower bound to hold. This α_{LB} value is decreasing in the ratio $\frac{\mu_{D_s}}{\sigma_{D_s}}$, creating a looser bound. The α_{UB} value is decreasing in the ratio $\frac{\mu_{D_s}}{\sigma_{D_s}}$, creating a tighter bound.

Thus I have found the optimal advance sales inventory X_a^* as a function of Q to be:

$$X_a^*(Q) = \begin{cases} Q, & \text{if } p_s = H \text{ and } [L(1-\alpha) - c_a]Q \geq H[\Lambda_{D_s}(0) - \Lambda_{D_s}(Q)]; \\ 0, & \text{if } p_s = H, \text{ otherwise;} \\ Q, & \text{if } p_s = L \text{ and } (H-L)\alpha \geq c_a \\ 0, & \text{if } p_s = L, \text{ otherwise.} \end{cases} \tag{3–38}$$

3.3.3 Optimal Order Policy (Q^*, X_a^*)

To find the optimal order policy (Q^*, X_a^*), I will use the variable substitution method common in the price-dependent newsvendor literature (see Petruzzi and Dada (1999) [33]). I will replace Q in the expected profit expression with the solution for $Q^*(X_a)$ found in section 3.3.1. I will then use the first order condition of $E[\Pi(Q^*)]$ to solve for X_a^* .

For the case when $p_s = L$, if conditions hold for $Q^*(X_a, L) = M$, I find that $E[\Pi(Q^*(L))]$ is linearly decreasing in X_a. Thus $X_a^*(L) = 0$ and I do not advance sell. If conditions hold for $Q^*(X_a, L) = 0$, then I know $X_a^*(L) = 0$ since $X_a \leq Q \leq M$. Thus, I conclude that if $p_s = L$ it is never optimal to advance sell. To determine the optimal inventory policy, I compare the profit earned from a $(Q, X_a) = (M, 0)$ policy with that from a $(0, 0)$ policy. Clearly, for the $p_s = L$ case, I have maximum profit with the $(M, 0)$ policy.

Theorem 5. *For $p_s = L$ and Bernoulli customer valuations, the optimal order policy is:*

$$(Q^*(L), X_a^*(L)) = (M, 0) \tag{3–39}$$

(See Appendix A for the proof.)

Thus, when the spot price is low, it is optimal to order for the entire market but not offer any advance sales.

For the case when $p_s = H$ I again find an extreme point solution for $X_a^*(H)$. However, since $Q^*(X_a, H)$ is a function of X_a (refer to equation 3–29) and $Q \leq M$, I do not have an upper bound of $X_a \leq Q$, but rather $X_a \leq M - \frac{\alpha k^2}{(1-\alpha)}$, where I define $k = \sqrt{2} erf^{-1}(2(\frac{H-c}{H}) - 1)$ as a constant (here, erf is the error function). Thus, I have two possible order policies (Q, X_a) for $p_s = H$: $(M, M - \frac{\alpha k^2}{(1-\alpha)})$ and $(F_{D_s}^{-1}(\frac{H-c}{H}), 0)$. The optimal policy is determined from the following condition.

Theorem 6. *For $p_s = H$ and Bernoulli customer valuations, the optimal order policy is:*

$$(Q^*(H), X_a^*(H)) = \begin{cases} (M, M - \frac{\alpha k^2}{(1-\alpha)}), & \textit{if } L(1-\alpha) + c \geq c_a, \\ (F_{D_s}^{-1}(\frac{H-c}{H}), 0), & \textit{otherwise.} \end{cases} \tag{3–40}$$

(See Appendix A for the proof.)

Therefore, when the spot price is high, if the advance sales cost is relatively low, it is optimal to order for the entire market reserve almost all of this inventory for advance selling. Otherwise, I order the standard newsvendor quantity and do not offer any advance sales.

I can write the optimal order policy as follows.

$$(Q^*, X_a^*) \;=\; \begin{cases} (M, M - \frac{\alpha k^2}{(1-\alpha)}), & \text{if } p_s = H \text{ and } L(1-\alpha) + c \geq c_a \\ (F_{D_s}^{-1}(\frac{H-c}{H}), 0), & \text{if } p_s = H, \text{ otherwise,} \\ (M, 0), & \text{if } p_s = L. \end{cases} \tag{3–41}$$

3.3.4 Optimal Pricing Strategy (p_a^*, p_s^*)

From my utility analysis of the customer's decision of whether or not to advance purchase, I concluded that a customer will advance purchase *iff* $p_a \leq \hat{p}_a$ (refer to equation 3–9). Therefore, since $\hat{p}_a$ is the maximum price I can offer in the advance sales period in order to earn X_a sales, the optimal advance sales price is $p_a^* = \hat{p}_a$.

Theorem 7. *The optimal advance sales price p_a^* is equivalent to the maximum advance sales price $\hat{p}_a$. For Bernoulli customer valuations, this optimal price is as follows.*

$$p_a^* = \hat{p}_a \quad = \quad \begin{cases} L + (H - L)\alpha, & \text{if } p_s = H; \\ L + (H - L)\alpha(\frac{M-Q}{M-X_a}), & \text{if } p_s = L. \end{cases} \tag{3–42}$$

To determine the optimal spot price p_s^*, I compare the expected profit from each price case (H or L) under the respective optimal order policy: $E[\Pi(Q^*(L), X_a^*(L))]$ and $E[\Pi(Q^*(H), X_a^*(H))]$. I have the following.

$$E[\Pi(Q^*(L), X_a^*(L))] \quad = \quad (L - c)M \tag{3–43}$$

$$\begin{aligned} E[\Pi(Q^*(H), X_a^*(H))] \quad = \quad & (L + (H - L)\alpha - c_a)X_a^*(H) \\ & + H((M - X_a^*(H))\alpha - \Lambda_{D_s}(F_{D_s}^{-1}(\frac{H - c}{H}))) \\ & - c(F_{D_s}^{-1}(\frac{H - c}{H}) + X_a^*(H)) \end{aligned} \tag{3–44}$$

I then have optimal spot price $p_s^* = H$ when $E[\Pi(Q^*(H), X_a^*(H))] > E[\Pi(Q^*(L), X_a^*(L))]$ and $p_s^* = L$ otherwise.

If I re-examine the optimal advance sales price p_a^* for $p_s = L$ given the optimal order policy $(Q^*(L), X_a^*(L)) = (M, 0)$, I have $p_a^* = L$. That is, if the spot price is $p_s = L$, then the advance sales price will also be $p_a = L$. This explains the previous result for $X_a^*(L) = 0$ (see 3–39). That is, if the spot price is $p_s = L$ then I do not advance sell since the probability of customers spot purchasing is 1 (see 3–14) and there is no extra revenue to be earned from advance sales ($p_a = p_s = L$). Thus I can write the optimal pricing policy as follows.

$$(p_a^*, p_s^*) = \begin{cases} (L + (H-L)\alpha, H), & \text{if } E[\Pi(Q^*(H), X_a^*(H))] \\ & > E[\Pi(Q^*(L), X_a^*(L))], \\ (L, L), & \text{otherwise.} \end{cases} \tag{3–45}$$

Where the optimal advance sales price p_a^* when $p_s = H$ is simply the customer's expected future valuation, $E[V]$, as defined in equation 3–22. Thus, if the expected profit from offering a high spot price is higher than the expected profit from offering a low spot price, the optimal pricing strategy is a high spot price and an advance sales price equal to the expected customer valuation. If the expected profit from a low spot price is higher, then the optimal pricing strategy is to offer a low price in both periods.

3.4 Numerical Experiments

I perform several numerical experiments to analyze the sensitivity of my analytical results to the customer valuation parameters α, H, and L, and to better understand the behavior of expected profit in the advance sales inventory decision, X_a. In these experiments, I use the variable substitution for $Q^*(X_a, H)$ to maximize $E[\Pi(Q^*(X_a, H))]$ by setting X_a as the decision variable. Finding the optimal advance sales inventory then determines the optimal value of $Q^*(X_a, H)$ (which is a function of $X_a^*(H)$). Thus, I focus on values for X_a^*, Q^*, and expected profit $E[\Pi]$ for $p_s = H$.

I perform a sensitivity analysis on the effect of the valuation probability α and the spread between the high and low valuation levels ($H - L$ spread) on the optimal values Q^*, X_a^*, and the expected profit. I also examine the effect on the percent of advance sales inventory ($\%AdvInv = X_a^*/Q^*$).

I assume the following parameters to be constant: advance sales cost, $c_a = 5$, unit cost, $c = 3$, and market size, $M = 100$. I vary the values of the valuation probability α between 0.1 and 0.9. I initially set $H = 80$ and $L = 20$. This yields the pricing policy $p_s = H = 80$ and $p_a = \hat{p}_a = 62$. I then vary the $H - L$ spread such that the advance sales price $p_a = \hat{p}_a$ remains fixed at this value.

α	H	L	p^a	Q(H)	Xa(H)	Xs(H)	% Adv Inv	E[Π(H)]
0.1	80	20	26	100.00	99.65	0.35	99.65%	1795.20
0.2	80	20	32	100.00	99.21	0.79	99.21%	2390.90
0.3	80	20	38	100.00	98.64	0.36	99.641%	2987.100
0.4	80	20	44	100.00	97.89	2.11	97.887%	3584.400
0.5	80	20	50	100.00	96.83	3.17	96.830%	4183.100
0.6	80	20	56	100.00	95.25	4.76	95.245%	4784.500
0.7	80	20	62	100.00	92.60	7.40	92.603%	5391.100
0.72	80	20	63.2	100.00	91.85	8.15	91.85%	5513.60
0.72125	80	20	63.275	100.00	91.20	8.80	91.20%	5521.20
0.7215625	80	20	63.29375	80.00	0.00	80.00	0.00%	5526.70
0.721875	80	20	63.3125	80.17	0.00	80.17	0.00%	5529.10
0.725	80	20	63.5	80.45	0.00	80.45	0.00%	5553.30
0.75	80	20	65	82.71	0.00	82.71	0.00%	5746.70
0.8	80	20	68	87.12	0.00	87.12	0.00%	6133.80
0.9	80	20	74	95.34	0.00	95.34	0.00%	6910.40

Table 3-1. Sensitivity Analysis for Varying α Values

As the valuation probability α changes, I see the results shown in Table 3-1 for Q^*, X_a^*, $X_s = Q - X_a$, $E[\Pi]$, and $\%AdvInv = X_a^*/Q^*$. I can make several observations. As α increases, Q^* is constant while $X_a^* > 0$, then Q^* increases after $X_a^* = 0$. Also, as α increases, X_a^* decreases as it becomes more profitable to reserve spot sales when the probability of a high spot price purchase increases. I can also observe that as α increases, the expected profit increases. This is due to more spot sales, which have a higher potential revenue ($p_s = H > p_a$).

The behavior of $E[\Pi]$ in X_a can be seen in the graphs in Figure 3-2, corresponding to the α values between 0.6 and 0.8 from the table in Table 3-1. From these graphs I confirm the extreme point solution for X_a. This behavior implies a "go/no-go" decision for offering advance sales. I can see that there is some threshold α value above which advance sales are no longer profitable. That is, once customers have a high enough probability of spot purchasing, it is more profitable to face the risk of holding all inventory for spot sales since again I have $p_s = H > p_a$.

As the $H - L$ Spread changes, while keeping the advance sales price $p_a = \hat{p}_a$ the same, I see the results shown in Table 3-2 for Q^*, X_a^*, $X_s = Q - X_a$, $E[\Pi]$, and

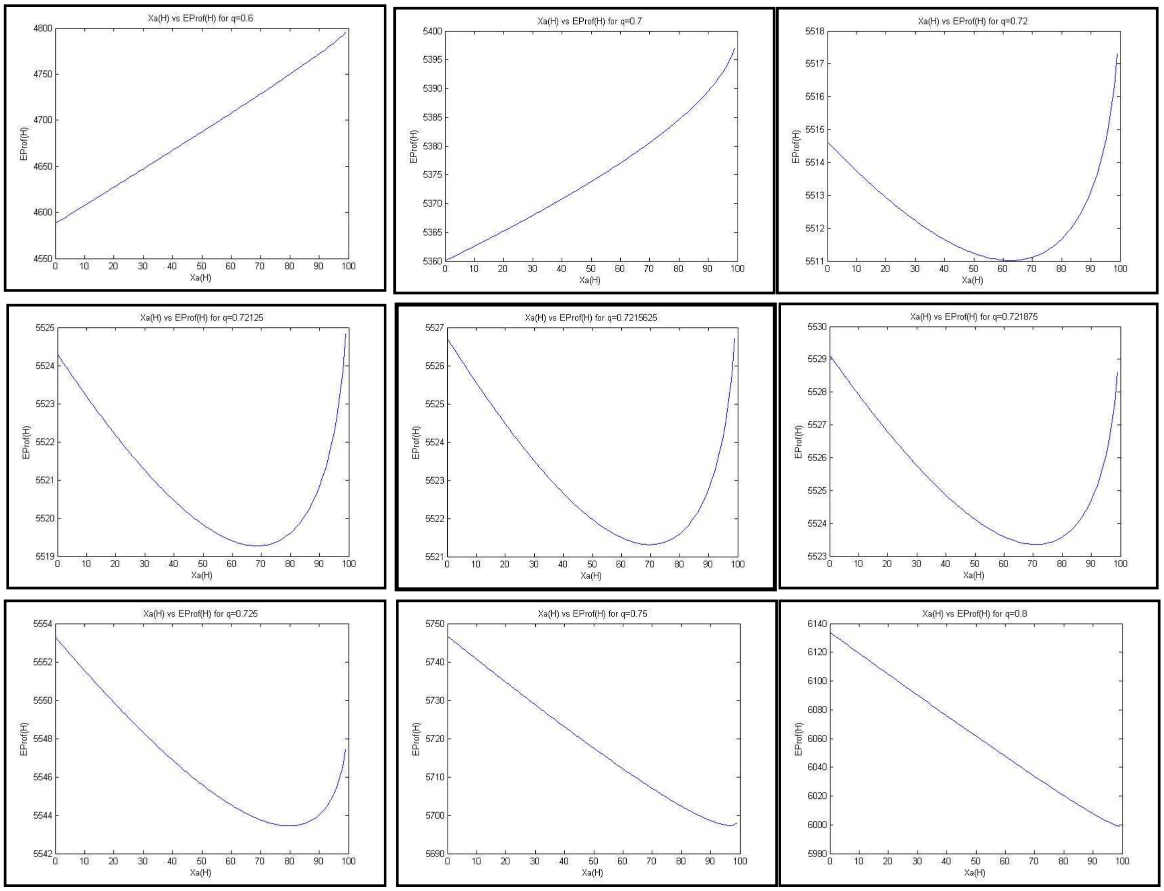

Figure 3-2. Graphs of X_a vs. $E[\Pi]$ for Varying α Values

$\%AdvInv = X_a^*/Q^*$. As the $H - L$ Spread decreases, I observe Q^* to be decreasing while $X_a = 0$ and then constant for $X_a = 0$. As the $H - L$ Spread decreases, I observe X_a^* to be increasing. X_a^* increases because the benefit of reserving spot sales decreases as the $H - L$ Spread decreases. That is, the spot price and advance sales price become close enough to outweigh the benefit of higher revenue for the uncertain spot sales. I can also observe that as the $H - L$ Spread decreases, the expected profit decreases. This is due to smaller price values.

The behavior of $E[\Pi]$ in X_a can be seen in the graphs in Figure 3-3, corresponding to the $H - L$ Spread values corresponding to the H values between 80.4 and 80.8 from the table in Table 3-2. From these graphs I can again confirm the extreme point solution for X_a implying a "go/no-go" decision for offering advance sales. I can see that there is some

α	H	L	H_L_Spread	p^a	Q(H)	Xa(H)	Xs(H)	% Adv Inv	E[Π(H)]
0.7	84.3	10	74.30	62	78.27	0.00	78.27	0.00%	5660.70
0.7	82.1	15	67.10	62	78.21	0.00	78.21	0.00%	5506.90
0.7	81.7	16	65.70	62	78.203	0.000	78.20	0.000%	5478.900
0.7	81.3	17	64.30	62	78.193	0.000	78.19	0.000%	5450.900
0.7	80.8	18	62.80	62	78.180	0.000	78.18	0.000%	5416.000
0.7	80.6	18.5	62.10	62	78.175	0.000	78.18	0.000%	5402.000
0.7	80.4	19	**61.40**	62	100.000	92.584	7.42	92.584%	5391.300
0.7	80	20	60.00	62	100.00	92.60	7.40	92.60%	5391.10
0.7	77.8	25	52.80	62	100.00	92.71	7.29	92.71%	5376.30
0.7	75.7	30	45.70	62	100.00	92.81	7.19	92.81%	5368.80
0.7	73.5	35	38.50	62	100.00	92.93	7.08	92.93%	5354.60
0.7	71.4	40	31.40	62	100.00	93.04	6.97	93.04%	5347.80
0.7	69.3	45	24.30	62	100.00	93.15	6.85	93.15%	5341.30
0.7	67.1	50	17.10	62	100.00	93.27	6.73	93.27%	5328.20
0.7	65	55	10.00	62	100.00	93.39	6.61	93.39%	5322.50
0.7	62.8	60	2.80	62	100.00	93.52	6.48	93.52%	5310.30

Table 3-2. Sensitivity Analysis for Varying $H - L$ Spread Values

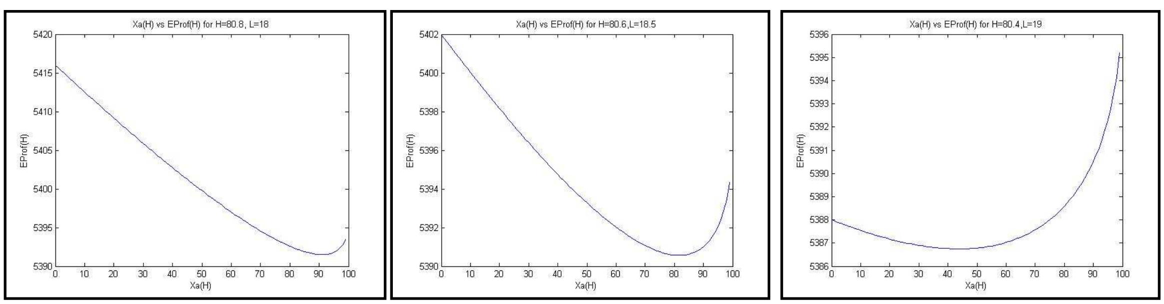

Figure 3-3. Graphs of X_a vs. $E[\Pi]$ for Varying $H - L$ Spread Values

threshold $H - L$ Spread value above which advance sales are no longer profitable. That is, once the $H - L$ spread is large enough, it is more profitable to face the risk of holding all inventory for spot sales since I have $p_s = H$ becoming increasingly larger than p_a.

I can compare the change in expected profit to the change in α values and $H - L$ Spread values to determine which parameter has the more sensitive effect. From the results shown in Table 3-3, it is clear that the maximum expected profit is more sensitive to the valuation probability α than to the $H - L$ Spread for the parameter values in the trials performed in Table 3-1 and Table 3-2.

	α	E[Π(H)]	*H-L Spread*	E[Π(H)]
max	0.9	6910.4	74.3	5660.7
min	0.1	1795.2	2.8	5310.3
% change	50.00%	44.43%	2553.57%	6.60%
sensitivity	**88.87%**		**0.26%**	

Table 3-3. Sensitivity Comparison of α vs. $H - L$ Spread Values

This may motivate a discussion on whether or not the firm can set these parameter values. For a Bernoulli distributed customer valuation, can a firm choose the probability of a high valuation, α? For this distribution, would α actually be some function of the parameters H and L? Can the firm set these values, and thus the $H - L$ Spread values, or are these determined by the market? In Section 3.5, I explore what would happen if the customer valuation distribution was Uniform instead of Bernoulli. In any case, if the firm is not able to effect the valuation distribution parameters, it can still determine the optimal order and pricing policy from the results of my model analysis.

3.5 An Extension

Let us consider the sensitivity of my results to the customer valuation distribution, specifically what happens if the customer valuation distribution is Uniform instead of Bernoulli. Assume that a customer's product valuation is distributed according to a continuous Uniform distribution between the low and high values (L, H). Now the probability that a customer will buy the product in the spot period is $Pr\{p_s \leq V\} = \frac{H-p_s}{H-L}$. I will use the expression $\bar{F}_V(p_s)$ to represent this spot purchase probability. Now I can consider any spot price $L \leq p_s < H$ since I have the following spot purchase probabilities.

$$Pr\{\text{spot purchase}\} = Pr\{p_s \leq V\} = \begin{cases} 0, & \text{if } p_s = H; \\ \frac{H-p_s}{H-L}, & \text{if } L < p_s < H; \\ 1, & \text{if } p_s = L. \end{cases} \quad (3\text{–}46)$$

I now have spot demand D_s distributed as follows.

$$D_s \sim Binomial(N_s = (M - X_a), \bar{F}_V(p_s)) \tag{3–47}$$

$$E[D_s] = (M - X_a)\bar{F}_V(p_s) \tag{3–48}$$

I will again approximate the Binomial spot demand with a Normal distribution. I define the mean and the standard deviation as follows.

$$\mu_{D_s} = (M - X_a)\bar{F}_V(p_s) \tag{3–49}$$

$$\sigma_{D_s} = \sqrt{(M - X_a)\bar{F}_V(p_s)F_V(p_s)} \tag{3–50}$$

I can then calculate the expected probability of available inventory β as follows.

$$E[\beta] = \frac{E[\min(Q - X_a, D_s)]}{E[D_s]} \tag{3–51}$$

$$= 1 - \frac{\Lambda_{D_s}(Q - X_a)}{(M - X_a)\bar{F}_V(p_s)} \tag{3–52}$$

Where Λ_{D_s} is the loss function $\int_{Q-X_a}^{\infty}(t - Q + X_a)f_{D_s}(t)dt$ and $f_{D_s}(t)$ is the Normal *pdf* with μ_{D_s} and σ_{D_s} as defined in 3–49 and 3–50.

I can now calculate the components of the customer's utility evaluation for an advance sales purchase, $E[U_a]$ and $E[U_s]$ as follows.

$$E[V] = \frac{H + L}{2} \tag{3–53}$$

$$E[(V - p_s)^+] = \int_{p_s}^{H}(t - p_s)f_V(t)dt \tag{3–54}$$

$$= \frac{1}{H - L}\int_{p_s}^{H}(t - p_s)dt \tag{3–55}$$

$$= \frac{(H - p_s)^2}{2(H - L)} \tag{3–56}$$

Where $E[V]$ and $f_V(t)$ are calculated from the continuous Uniform distribution.

The maximum advance sales price $\hat{p}_a$ is then.

$$\hat{p}_a = E[V] - E[(V-p_s)^+]E[\beta] \quad (3\text{–}57)$$

$$= \frac{H+L}{2} - \frac{(H-p_s)^2}{2(H-L)} + \frac{\Lambda_{D_s}(Q-X_a)(H-p_s)}{2(M-X_a)} \quad (3\text{–}58)$$

I can now write the expected profit expression as follows.

$$E[\Pi] = \left(\frac{H+L}{2} - \frac{(H-p_s)^2}{2(H-L)} + \frac{\Lambda_{D_s}(Q-X_a)(H-p_s)}{2(M-X_a)} - c_a\right)X_a + p_s\left((M-X_a)\frac{H-p_s}{H-L} - \Lambda_{D_s}(Q-X_a)\right) - cQ \quad (3\text{–}59)$$

I then solve for the optimal inventory quantity Q^* as a function of X_a.

Theorem 8. *For Uniform customer valuations, the expected profit $E[\Pi]$ is concave in Q and the optimal order quantity Q^* for a given X_a is:*

$$Q^* = F_{D_s}^{-1}\left(\frac{p_s - c - \frac{X_a(H-p_s)}{2(M-X_a)}}{p_s - \frac{X_a(H-p_s)}{2(M-X_a)}}\right) + X_a \quad (3\text{–}60)$$

(See Appendix A for the proof.)

I find a result that resembles the newsvendor model I found previously (see 3.3.1), except this time with a smaller value. That is, $F_{D_s}^{-1}\left(\frac{p_s - c - \frac{X_a(H-p_s)}{2(M-X_a)}}{p_s - \frac{X_a(H-p_s)}{2(M-X_a)}}\right) < F_{D_s}^{-1}\left(\frac{H-c}{H}\right)$.

To find X_a^*, I perform similar numerical experiments as done in Section 3.4. In these experiments, I again use the variable substitution for Q^* to maximize $E[\Pi(Q^*)]$ by setting X_a^* as the decision variable. For these trials, I fix the advance sales cost, $c_a = 5$, unit cost, $c = 3$, and market size, $M = 100$. Then, since I do not know the spot price p_s, I set $H = 80$ and $L = 20$ and compare the results for the spot price values $L < p_s < H$.

As the spot price p_s changes, for $H = 80$ and $L = 20$, I see the results shown in Table 3-4 for Q^*, X_a^*, $X_s = Q - X_a$, $E[\Pi]$, and $\%AdvInv = X_a^*/Q^*$. As p_s decreases, Q^* is decreasing while $X_a^* > 0$, then Q^* increases after $X_a^* = 0$. Also, as p_s decreases, X_a^* decreases and the expected profit decreases. The values of X_a^* and $E[\Pi]$ are higher with larger p_s values since a larger spot price creates a larger advance sales price $p_a = \hat{p}_a$.

ps	Fv	p^a	Q	Xa	Xs	% Adv Inv	E[Π]
79	0.02	50.02	99.33	99.32	0.01	1.00	4171.20
69	0.18	49.99	92.33	92.27	0.06	1.00	3875.20
59	0.35	49.08	85.59	84.20	1.39	0.98	3536.40
49	0.52	45.92	81.43	74.79	6.64	0.92	3141.30
39	*0.68*	*40.26*	*80.96*	*63.71*	*17.25*	*0.79*	*2676.00*
29	*0.85*	*28.37*	89.51	*0.00*	89.51	0.00	2191.40
21	*0.98*	*21.02*	99.70	*0.00*	99.70	0.00	1763.90

Table 3-4. Sensitivity Analysis for Varying p_s Values ($V \sim Uniform(20, 80)$)

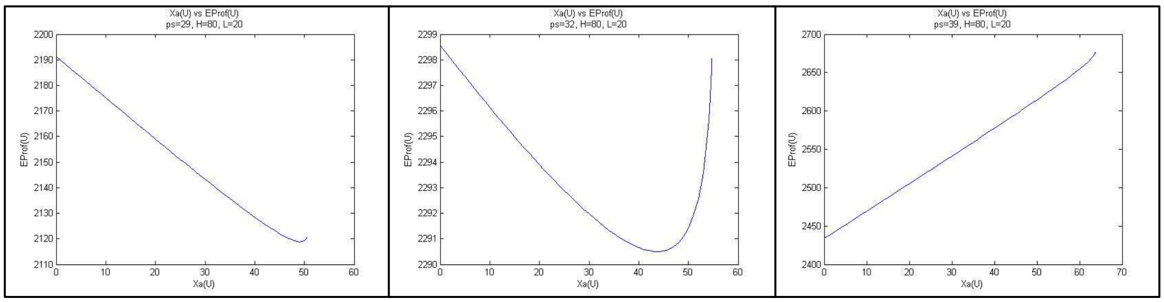

Figure 3-4. Graph of X_a vs. $E[\Pi]$ ($V \sim Uniform(20, 80)$)

I can conclude that the largest profit is attained when p_s is maximum. Thus I have an optimal pricing policy of $(p_s^* \approx H, p_a^* = \hat{p}_a(p_s^*))$ and an optimal order policy of $(Q^* = F_{D_s}^{-1}(\frac{H-c}{H}) + X_a^*, X_a^*)$. That is, I use the high p_s value to drive up the value of p_a. Since the probability of spot purchase, $\bar{F}_V(p_s)$, becomes very low with such a high p_s value, I advance sell to everyone at p_a.

The behavior of $E[\Pi]$ in X_a can be seen in the graphs in Figure 3-4. From these graphs I observe the extreme point solution for X_a. I can see that there is some threshold p_s value below which advance sales are no longer profitable. I find similar results for various H and L values.

Let us try to compare these results to the case with Bernoulli customer valuations. In the first set of trials in Section 3.4, shown in Table 3-1,I set $H = 80$ and $L = 20$ and vary the α values (where α is the probability of a spot purchase when $p_s = H$). In the case of Uniform customer valuations, the probability of a spot purchase, $\bar{F}_V(p_s)$ is a function of

p_s. In order to compare my results with the Bernoulli customer valuation case, I must set $p_s \approx H$, which yields a very low value for $\bar{F}_V(p_s)$. Therefore, I can compare the first row in Table 3-1 with the first row in Table 3-4. I can observe the following. The advance sales price $p_a = \hat{p}_a$ is higher in the Uniform case than in the Bernoulli case. The inventory levels Q and X_a are both high. The expected profit $E[\Pi]$ is higher in the Uniform case than in the Bernoulli case. Thus I can observe that for $p_s = H$ and $H = 80$, $L = 20$, and α or $\bar{F}_V(p_s)$ very low, I have similar inventory policies but higher profit in the Uniform case.

To further compare these two customer valuation distributions, I repeat the trials for various $H - L$ Spread values, this time with the same spot price and spot purchase probability values. Since I examine the case when $p_s = H$ for Bernoulli customer valuations, I want to only compare the Uniform customer valuation trials for $p_s \approx H$, or $p_s = H - 1$. Then, since the probability of a spot purchase for Uniform customer valuations, $\bar{F}_V(p_s)$, is affected by the price p_s, I will set the Bernoulli spot purchase probability $\alpha = \bar{F}_V(p_s)$ for a fair comparison. I thus compare the trials shown in Table 3-5.

I observe that as the $H - L$ Spread decreases, the optimal advance sales inventory X_a^* decreases in both cases. The optimal order quantity Q^* is decreasing in the Uniform case, but constant in the Bernoulli case. This difference can be explained by the difference in the Q^* expression described earlier, where I have Q^* for the Uniform case smaller than Q^* for the Bernoulli case (see Theorem 8). The main difference between these valuation cases is in the behavior of the expected profit. In the Uniform case, I have expected profit $E[\Pi]$ decreasing as the $H - L$ Spread decreases, whereas in the Bernoulli case, $E[\Pi]$ is increasing. This difference is attributed to the calculation of $p_a = \hat{p}_a$. Since X_a is high in both cases, most of the profit comes from advance sales, and thus is affected by the advance sales price p_a. In the Uniform case, the advance sales price, when the spot price is high ($p_s \approx H$), is the expected valuation $E[V] = \frac{H+L}{2}$ which is constant. In the Bernoulli case, the advance sales price is also the expected valuation, but in this case $E[V] = H\alpha + L(1 - \alpha)$ is a function of α. Therefore, I have expected profit increasing

Uniform Customer Valuations										
ps=H-1	Fv	H	L	H_L_Spread	p^a	Q	Xa	Xs	% Adv Inv	E[Π]
89	0.01	90	10	80.00	50.02	99.41	99.40	0.01	99.99%	4174.60
84	0.01	85	15	70.00	50.02	99.37	99.36	0.01	99.99%	4172.90
79	0.02	80	20	60.00	50.02	99.33	99.32	0.01	99.99%	4171.20
74	0.02	75	25	50.00	50.02	99.28	99.27	0.01	99.99%	4169.10
69	0.03	70	30	40.00	50.02	99.23	99.22	0.01	99.99%	4167.00
64	0.03	65	35	30.00	50.02	99.16	99.15	0.01	99.99%	4164.00
59	0.05	60	40	20.00	50.03	99.10	99.08	0.02	99.98%	4161.00
54	0.10	55	45	10.00	50.03	99.01	98.99	0.02	99.98%	4157.20

Bernoulli Customer Valuations										
ps=H	α	H	L	H_L_Spread	p^a	Q(H)	Xa(H)	Xs(H)	% Adv Inv	E[Π(H)]
90	0.01	90	10	80.00	11	100.00	99.96	0.04	99.96%	279.81
85	0.01	85	15	70.00	16	100.00	99.96	0.04	99.96%	769.65
80	0.02	80	20	60.00	21	100.00	99.93	0.07	99.93%	1319.00
75	0.02	75	25	50.00	26	100.00	99.93	0.07	99.93%	1798.70
70	0.03	70	30	40.00	31	100.00	99.91	0.09	99.91%	2317.70
65	0.03	65	35	30.00	36	100.00	99.91	0.09	99.91%	2787.40
60	0.05	60	40	20.00	41	100.00	99.85	0.15	99.85%	3295.20
55	0.10	55	45	10.00	46	100.00	99.71	0.29	99.71%	3789.70

Table 3-5. Sensitivity Analysis for Varying $H - L$ Spread Values for Uniform and Bernoulli Customer Valuations

as the $H - L$ Spread decreases for the Bernoulli case because the advance sales price is increasing.

Thus, if the customer valuation distribution is different, the structural results hold (extreme point solution for X_a, Q^* has a newsvendor component plus X_a), but the sensitivity to the valuation distribution parameters may vary.

Please refer to Chapter 6 for the related conclusions and future research extensions.

CHAPTER 4
MULTI-GENERATION PRICING AND TIMING DECISIONS IN NEW PRODUCT DEVELOPMENT

4.1 Introduction and Motivation

The analytic model introduced here utilizes a two generation framework and incorporates elements from dynamic pricing and time-to-market bodies of literature. my key decision variables are the dynamic pricing strategy for each generation of products and the optimal time to introduce the second generation of products. A main factor influencing these decisions is the anticipated shape of the demand/sales curve for each generation of new products. For example, the product life cycle curve is often associated with the introduction, growth and decline of a product in the marketplace via some kind of diffusion process. Conversely, a common assumption in the literature addressing the optimal time-to-market for new product introductions is that price (and consequently sales) is static for both old and new generations of products. Other factors included in the model are the dynamic unit costs for each generation as well as the development costs associated with the second generation of new products.

I utilize optimal control methodologies to characterize the optimal pricing strategy for both generations and the timing for the introduction of the second generation. Analytic results for specific cases reflecting different assumptions concerning the demand process are developed which directly link the price and timing decisions. When sales are dependent on price only (i.e. no diffusion effects), the optimal policy is to introduce only the single most profitable generations of products in most situations. Specifically, I either introduce the second generation at the start of the planning horizon, or not at all (i.e. a now or never policy). When sales are dependent on diffusion only (i.e. not a direct function of price), then the optimal timing of introduction of the second generation of products is dependent on the length of the planning horizon. I derive a threshold value such that if the length of the planning horizon is smaller than the threshold, then a single generation solution is optimal. If the length of the planning horizon exceeds the threshold value, then it is optimal to introduce both generations to market in a sequential manner. In this case, the optimal time to market for the second generation is dependent on the price, cost, and diffusion parameters for both generations of products.

Please refer to the Chapter 2 for a review of the related literature.

4.2 Model

My model considers two generations of a new product: Generation 1 and Generation 2. I assume that sales for Generation 1 start at time 0, while the market entry time for Generation 2 is a decision variable in the model. I want to determine the optimal dynamic price of each generation, $p_1^*(t)$ and $p_2^*(t)$, as well as the optimal time to introduce Generation 2 to the market, t_m^*.

I consider a single rollover scenario where the sales for Generation 1 will stop once Generation 2 is introduced to the market. Both price and cost are dynamic variables in my model. I also include a fixed cost for introducing Generation 2 to the market, c_{t_m} which is a one-time fixed cost incurred if and when I introduce Generation 2 to the market. This cost may be attributed to development needs or marketing expenses. I assume that the firm has a fixed time horizon, T. Although I consider the time horizon to be exogenous, I will discuss later the effect of its value on the optimal introduction time of Generation 2.

Thus, I have the following decision variables:

$p_1(t)$	unit price at time t of current Generation 1
$p_2(t)$	unit price at time t of current Generation 2
t_m	time at which Generation 2 is introduced to the market

And I define the following notation:

T	length of the planning horizon
$c_1(t)$	unit cost at time t of current Generation 1
$c_2(t)$	unit cost at time t of current Generation 2
c_{t_m}	fixed cost of introducing Generation 2 to the market at time t_m
$\dot{x}_1(t)$	sales rate at time t of Generation 1
$\dot{x}_2(t)$	sales rate at time t of Generation 2
$x_1(t)$	cumulative sales in the time interval $(0, t)$ of Generation 1
$x_2(t)$	cumulative sales in the time interval $(0, t)$ of Generation 2
$\lambda_1(t)$	marginal value of selling one unit of Generation 1
$\lambda_2(t)$	marginal value of selling one unit of Generation 2
$\lambda_3(t)$	marginal value of introducing Generation 2 to the market
$Y(t)$	binary indicator variable corresponding to time-to-market $Y(t) = \begin{cases} 0, & t < t_m, \\ 1, & t \geq t_m. \end{cases}$
$v(t)$	impulse variable corresponding to time-to-market $v(t) = \begin{cases} 1, & t = t_m, \\ 0, & \text{otherwise.} \end{cases}$
δ	the dirac delta function

My objective is to determine the optimal prices for the two generations and the optimal introduction time for the second generation such that total profit is maximized over the time horizon. The objective function is stated mathematically in Equation 4–1. I define total profit as the net revenue for both generations earned over the time horizon, minus the cost of introduction to market for the second generation. Note that I ignore discounting in my model. This is done for clarity purposes. I desire to focus on the optimal pricing and timing decisions derived from a simple expression of profit. I have conducted my analysis for the discounting case and find similar results. Thus, for the scope of this chapter I will assume there is no discounting, although my results hold for the discounting case.

$$\text{Max} \quad \int_0^T (\dot{x}_1(t)(p_1(t) - c_1(t)) + \dot{x}_2(t)(p_2(t) - c_2(t)))dt - c_{t_m} v(t_m) \tag{4–1}$$

$$\text{s.t.} \quad \lambda_1(t) : \dot{x}_1(t) = f(x_1(t), p_1(t))(1 - Y(t)) \tag{4–2}$$

$$\lambda_2(t) : \dot{x}_2(t) = g(x_2(t), p_2(t))Y(t) \tag{4–3}$$

$$\lambda_3(t) : \dot{Y}(t) = \delta(t - t_m)v(t) \tag{4–4}$$

The constraint in Equation 4–2 defines the sales rate for Generation 1. Since my binary indicator variable $Y(t)$ is initially 0, the sales rate for Generation 1 is positive for $t < t_m$ and 0 thereafter. The marginal value of selling one more unit of Generation 1, $\lambda_1(t)$, will later be derived from this constraint. Likewise, the constraint in Equation 4–3 defines the sales rate for the second generation. Based on the binary indicator variable $Y(t)$, the sales rate for the second generation is 0 for $t < t_m$ and positive thereafter. The marginal value of selling one more unit of Generation 2, $\lambda_2(t)$, will later be derived from this constraint. The third constraint in Equation 4–4 defines the rate of change of the binary indicator variable in terms of the impulse variable and the dirac delta function. The marginal value of introducing Generation 2 to the market, $\lambda_3(t)$, will later be derived from this constraint.

I solve this model using Optimal Control Theory, which is a non-linear optimization methodology utilized for dynamic economic problems. For a summary of the methodology and applications, refer to Sethi and Thompson (2000) for details. For my model, the control variables are the prices for each generation: $p_1(t)$ and $p_2(t)$. The state variables are the cumulative sales for each generation: $x_1(t)$ and $x_2(t)$. The adjoint variables are the three marginal values defined in the problem constraints: $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$. The impulse control variable, $v(t_m)$, is used to find the optimal time to market, t_m^*. I define the Hamiltonian and Impulse Hamiltonian as follows (henceforth, I remove the time variable for clarity).

$$H = (p_1 - c_1 + \lambda_1) f(x_1, p_1)(1 - Y) + (p_2 - c_2 + \lambda_2) g(x_2, p_2) Y \tag{4–5}$$

$$H^I = (\lambda_3 - c_{t_m}) v(t_m) \tag{4–6}$$

I derive the following expressions utilizing the necessary conditions for optimality. The first expressions below are for the adjoint variables, which for my model are defined as the rates of change for the marginal values associated with the cumulative sales for each generation $x_1(t)$ and $x_2(t)$.

$$\dot{\lambda}_1 = \frac{-\delta H}{\delta x_1} = -(p_1 - c_1 + \lambda_1) \frac{\delta f}{\delta x_1} (1 - Y) \tag{4–7}$$

$$= \begin{cases} -(p_1 - c_1 + \lambda_1) \frac{\delta f}{\delta x_1}, & t < t_m, \\ 0, & t \geq t_m. \end{cases} \tag{4–8}$$

$$\dot{\lambda}_2 = \frac{-\delta H}{\delta x_2} = -(p_2 - c_2 + \lambda_2) \frac{\delta g}{\delta x_2} Y \tag{4–9}$$

$$= \begin{cases} 0, & t < t_m, \\ -(p_2 - c_2 + \lambda_2) \frac{\delta g}{\delta x_2}, & t \geq t_m. \end{cases} \tag{4–10}$$

$$\dot{\lambda}_3 = \frac{-\delta H}{\delta Y} = (p_1 - c_1 + \lambda_1) f(x_1, p_1) - (p_2 - c_2 + \lambda_2) g(x_2, p_2) \tag{4–11}$$

Note that since I do not have any salvage value in my objective function expression, I have $\lambda_1(T) = 0$, $\lambda_2(T) = 0$, and $\lambda_3(T) = 0$.

The next expressions correspond to the continuous control variables, which in my model are the optimal price for each generation.

$$p_1^* : \qquad \frac{\delta H}{\delta p_1} = 0 \tag{4–12}$$

$$f(x_1, p_1) + (p_1 - c_1 + \lambda_1)\frac{\delta f}{\delta p_1} = 0 \tag{4–13}$$

$$p_2^* : \qquad \frac{\delta H}{\delta p_2} = 0 \tag{4–14}$$

$$g(x_2, p_2) + (p_2 - c_2 + \lambda_2)\frac{\delta g}{\delta p_2} = 0 \tag{4–15}$$

Notice that the optimal dynamic prices depend on the functional forms of the sales rates for each generation.

The next expressions correspond to the impulse control variable, in my model the optimal time to market for Generation 2.

$$v^*(t_m) : \qquad \frac{\delta H^I}{\delta v(t_m)} = 0 \tag{4–16}$$

$$\frac{\delta H^I}{\delta v} = \lambda_3 - c_{t_m} \tag{4–17}$$

$$v^*(t_m) = \begin{cases} 1, & \lambda_3 \geq c_{t_m}, \\ 0, & \lambda_3 < c_{t_m}. \end{cases} \tag{4–18}$$

$$t_m^* : \qquad v^*(t_m^*) = 1 \Rightarrow \lambda_3(t_m^*) \geq c_{t_m^*} \tag{4–19}$$

I can interpret this expression as follows: I only introduce Generation 2 if the marginal value outweighs the introduction to market cost. Notice that the optimal time to market will depend on the shape of $\lambda_3(t_m)$ and its value relative to the fixed costs.

4.3 Analysis

To complete my analysis, I assume the following functional forms for $f(x_1, p)$ and $g(x_2, p)$, the sales rates for Generation 1 and 2, respectively. These functional forms capture the inverse relationship between price and sales and the Bass Model behavior of sales diffusion considering innovators and imitators. Specifically, I assume that the forms are linear additive functions of price and sales diffusion. (Refer to Padmanabhan and Bass (1993) [17], Teng and Thompson (1996) [16], Mahajan, Muller, and Bass (1990) [34], and Kalish (1983) [12] for similar functions.)

$$f(x_1, p_1) = a_0 - a_1 p_1 + a_2[\phi(M_1 - x_1) + \frac{\psi}{M_1}(M_1 - x_1)x_1] \quad (4\text{–}20)$$

$$g(x_2, p_2) = b_0 - b_1 p_2 + b_2[\phi(M_2 - x_2) + \frac{\psi}{M_2}(M_2 - x_2)x_2] \quad (4\text{–}21)$$

Furthermore, I define the following variables associated with the sales functions:

a_0, a_1, a_2	positive constants
b_0, b_1, b_2	positive constants
M_1	market size of Generation 1
M_2	market size of Generation 2
ϕ	coefficient of innovation
ψ	coefficient of imitation

In the following subsections, I analyze special cases of these functions.

4.3.1 CASE 1: Price Effect Only, No Diffusion Effect

First I consider the isolated effect of price on sales. Let us assume that $a_2 = b_2 = 0$ such that there are no diffusion effects impacting the sales function. The sales rate functions can then be written as follows:

$$f(x_1, p_1) = a_0 - a_1 p_1 \quad (4\text{–}22)$$

$$g(x_2, p_2) = b_0 - b_1 p_2 \quad (4\text{–}23)$$

Theorem 1: Optimal Prices for CASE 1 *When the sales function includes prices effect only, the optimal price p_1^* for Generation 1 and p_2^* for Generation 2 are:*

$$p_1^* = \frac{1}{2}(\frac{a_0}{a_1} + c_1) \quad (4\text{–}24)$$

$$p_2^* = \frac{1}{2}(\frac{b_0}{b_1} + c_2) \quad (4\text{–}25)$$

Note that the optimal prices are linear additive functions of the unit costs. Therefore, the price changes at a same rate over time as the corresponding cost for that generation. I can also observe that price is decreasing convex in the price weight constants; that is p_1 is decreasing convex in a_1, and p_2 is decreasing convex in b_1. As the weight of price increases, the same price will have a larger (negative) effect on sales (refer to Eqns 4.3 and 4.5). Therefore, the optimal price decreases in order to maximize the tradeoff between the profit margin and sales. I may also note that when unit costs are constant, then the optimal prices are also constant.

Theorem 2: Optimal Time to Market for CASE 1 *When the sales function includes price effect only, the optimal time to market t_m^* is:*

$$t_m^* : -\int_{t_m}^{T} (K_1 - K_2)dt \geq c_{t_m} \tag{4–26}$$

Where I define $K_1 = (p_1 - c_1)(a_0 - a_1 p_1)$ and $K_2 = (p_2 - c_2)(b_0 - b_1 p_2)$ as the profit rate for each generation.

Note that if $K_1 > K_2 \forall t$, then $\lambda_3(t_m) < 0$ and therefore never greater than $c_{t_m} > 0$, so I never introduce Generation 2. If, however, $K_2 > K_1 \forall t$, then $\lambda_3(t_m)$ is decreasing in t_m. In this situation, if $\lambda_3(0) > c_{t_m}$, I introduce Generation 2 immediately. But if $K_2 > K_1 \forall t$ and $\lambda_3(0) < c_{t_m}$, then I never introduce Generation 2.

A two generation solution may be optimal when unit costs are changing over the planning horizon. Consider the case where the first generation of products is fairly mature such that the unit cost(c_1)is constant, while the unit cost for the second generation of products is decreasing over time. Suppose also that the product margins are such that $K_2 < K_1$ initially, $K_2 = K_1$ at some point t_s^* during the planning horizon, and $K_2 > K_1$ at the end of the planning horizon. The shape of $\lambda_3(t)$ in this situation is such that it starts out fairly low, increases to a peak at $t = t_s^*$, and then decreases to zero at t=T. If $\lambda_3(t_s^*) > c_{t_m}$, then the optimal time to market for the second generation is some time before t_s^*. This result concurs with Carrillo and Franza (2004) who find that the optimal time-to-market can occur before the marginal value of sales of the new product exceed those of the old product.

If unit costs are constant over the entire planning horizon, then I have $\lambda_3(t_m) = (K_1 - K_2)(T - t_m)$. In Figure 4-1, I illustrate this behavior of $\lambda_3(t_m)$. I can see clearly, that when $K_2 > K_1$, because $\lambda_3(t_m)$ is decreasing, I have a now or never optimal time to market depending on the value of $\lambda_3(0)$ relative to c_{t_m}.

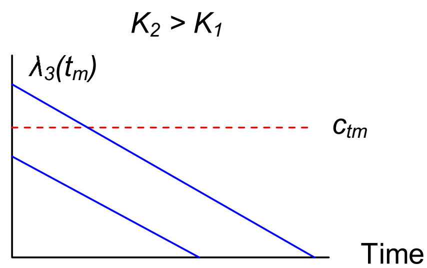

Figure 4-1. Evaluating $\lambda_3(t_m) \geq c_{t_m}$ when $K_2 > K_1$ and costs are constant.

Corollary 1: Optimal Time to Market for CASE 1 Under Constant Costs *When the unit costs are constant, the optimal time to market is as follows:*

$$t_m^* = \begin{cases} 0 \text{ (now)}, & (K_2 - K_1)T \geq c_{t_m}, \\ \text{never}, & (K_2 - K_1)T < c_{t_m} \text{ or } K_1 > K_2. \end{cases} \tag{4–27}$$

Keeping in mind that K_1 and K_2 are functions of price and cost, I can observe that t_m^* is decreasing in c_1 and increasing in c_2. This is intuitive since a larger unit cost for Generation 1 will make it less profitable and therefore t_m^* will decrease, or shift, from "never" to "now", meaning only Generation 2 will be sold. Likewise, an increasing unit cost for Generation 2 will make it less profitable and therefore t_m^* will shift from "now" to "never", meaning only Generation 1 will be sold. It is clear that t_m^* is increasing in c_{t_m}. That is, t_m^* shifts from "now" to "never" if the cost of introducing Generation 2 is too high.

I also observe that t_m^* decreases in a_1 and increases in b_1. Similar to the sensitivity of price to these parameters, as the weight of price increases, the same price will have a larger (negative) effect on sales. Therefore, an increase in these parameters makes selling the corresponding generation less profitable due to a negative effect on sales and also an implicit decrease in price, and therefore the time each generation is sold on the market decreases. So as a_1 increases, t_m^* shifts from "never" to "now", and as b_1 increases, t_m^* shifts from "now" to "never".

Corollary 2: Optimal Time to Market for CASE 1 Under Constant Costs with Horizon Threshold *Given a horizon threshold defined as* $\bar{T} = \frac{c_{tm}}{(K_2 - K_1)}$, *the optimal* t_m^* *can be written as follows:*

$$t_m^* = \begin{cases} 0 \text{ (now)}, & T \geq \bar{T}, \\ \text{never}, & T < \bar{T} \text{ or } K_1 > K_2. \end{cases} \tag{4–28}$$

Here, $\bar{T}$ represents the tradeoff between the profit margin and the cost of introducing a second generation. I can observe that t_m^* is decreasing in T. That is, as the planning horizon increases, it is more profitable to sell Generation 2 only (given that $K_1 \leq K_2$). Note: If $K_1 = K_2$, then I have $\bar{T} = \infty$ and $T < \bar{T}$ will always be true. Thus, if I have equal generations, then I only sell Generation 1 and never introduce Generation 2. This seems intuitive; when I have equal generations, a positive introduction to market cost c_{t_m} and $\lambda_3(t_m)$ decreasing in t_m both make Generation 1 to be the more profitable single generation to be sold.

In summary, the key driver determining the optimal time to market for a linear price demand model is the unit margins for each of the generations. When I consider a price effect only on sales with constant unit costs, the optimal time to market for Generation 2 is either now or never. The optimal price in this situation is constant over the total planning horizon. This concurs with the literature for single-generation models with a price-only sales effect (see Kalish [12]).

4.3.2 CASE 2: Diffusion Effect Only, No Price Effect

I now consider the isolated effect of diffusion on sales. Let us assume that $a_0 = a_1 = 0$ and $b_0 = b_1 = 0$, yielding the following sales rate functions:

$$f(x_1, p_1) = a_2[\phi(M_1 - x_1) + \frac{\psi}{M_1}(M_1 - x_1)x_1] \quad (4\text{–}29)$$

$$g(x_2, p_2) = b_2[\phi(M_2 - x_2) + \frac{\psi}{M_2}(M_2 - x_2)x_2] \quad (4\text{–}30)$$

Theorem 3: Optimal Prices for CASE 2 *When the sales function includes diffusion effect only, the optimal prices p_1^* and p_2^* are equal to the maximum market prices $\hat{p}_1$ and $\hat{p}_2$, respectively.*

Theorem 4: Optimal Time to Market for CASE 2 *When the sales function includes diffusion effect only, the optimal time to market t_m^* is:*

$$t_m^* : [(p_2 - c_2)g(x_2(T - t_m)) - (p_1 - c_1)f(x_1(t_m))](T - t_m) \geq c_{t_m} \quad (4\text{–}31)$$

The above equation shows that the optimal t_m is such that the profit earned for the remaining time from introducing Generation 2 is greater than the cost of introducing Generation 2 to the market.

I am not able to find a closed form solution for t_m^*. However, I do investigate its sensitivity to other parameters. I find $\lambda_3(t_m)$ to be decreasing in p_1, c_2, and M_1, which yields a larger t_m value. This result implies that for larger p_1 and/or M_1 values, I want to sell Generation 1 for a longer time. Similarly, for larger c_2 values, I want to sell Generation 2 for a shorter time, and thus t_m is larger. I find opposite behaviors for the complimentary parameters: $\lambda_3(t_m)$ is increasing in p_2, c_1, and M_2. In this case, I want to sell Generation 2 for a longer time so t_m is smaller. I can observe, that although I do not consider the price effect on the sales rate in this section, the optimal prices p_1^* and p_2^* still impact the optimal time to market t_m^*.

In the Numerical Analysis section, I investigate the behavior of t_m^* under various parameter assumptions.

4.3.2.1 Time horizon threshold

Another significant parameter driving the optimal time to market is the length of the planning horizon. In studying the behavior of $\lambda_3(t_m)$, I notice that for small T values, $\lambda_3(0) > 0$ and is then generally decreasing. Consequently, it is optimal to sell only one generation if $\lambda_3(0) > c_{t_m}$. However, for larger T values, $\lambda_3(0) < 0$ and is then generally increasing. (Refer to Figure 4-2 for an illustration of this switching behavior in $\lambda_3(t_m)$.)

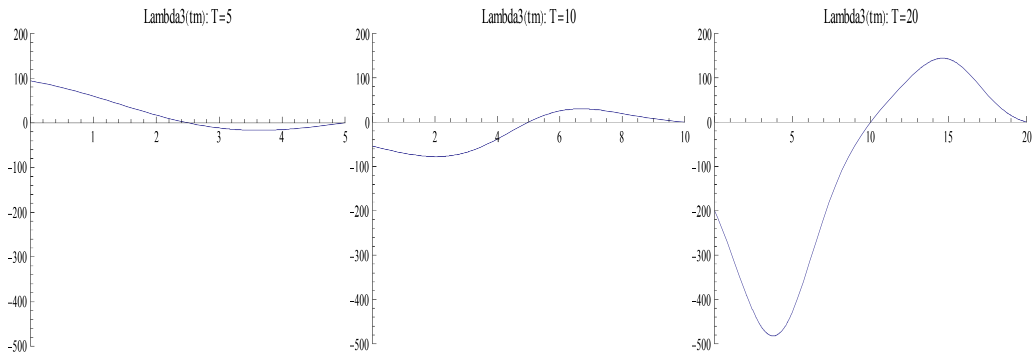

Figure 4-2. The switching behavior of $\lambda_3(t_m)$ as T increases.

I conclude that there exists a threshold value of T, which I call $\bar{T}$, after which there is a shift in the behavior and $\lambda_3(t_m)$. The explanation of this threshold value is somewhat intuitive. If the time horizon (T) is too short, then cumulative sales benefits cannot be fully realized. Specifically, the peak sales may not be reached. Thus it may be more profitable to sell only one generation. Whereas for larger T values, more cumulative sales can be accrued and thus it may be more profitable to introduce both generations. This threshold behavior reveals that although T is considered to be a fixed parameter, it plays an important role in the determination of t_m^*.

Theorem 5: Horizon Threshold for CASE 2 Under Negligible Introduction Cost *When the sales function includes diffusion effect only, and the introduction cost is negligible*

($c_{t_m} = 0$), a horizon threshold can be derived:

$$\bar{T} = \frac{\log\left(\frac{\begin{array}{l}(2a_2M_1(-c_1+p_1)\phi\psi + b_2M_2(c_2-p_2)(\phi+\psi)^2 \\ -(\phi+\psi)\sqrt{b_2M_2(c_2-p_2)(4a_2M_1(-c_1+p_1)\phi\psi + b_2M_2(c_2-p_2)(\phi+\psi)^2))}\end{array}}{2a_2M_1(c_1-p_1)\phi^2}\right)}{b_2(\phi+\psi)} \tag{4–32}$$

Corollary 3: Optimal Time to Market for CASE 3 with Horizon Threshold *Given the horizon threshold $\bar{T}$ defined in Theorem 5, the optimal time to market t_m^* can be written as follows:*

$$t_m^* \begin{cases} = 0 \text{ (now) or } = T \text{ (never)}, & T \leq \bar{T}, \\ > 0 \text{ (later)}, & T > \bar{T}. \end{cases} \tag{4–33}$$

Where, for the case where $t_m^* > 0$, I have t_m^* as defined in Theorem 4.

Therefore, the optimal time to market is defined relative to the planning horizon threshold. For relatively short planning horizons, a single generation is optimal. For longer planning horizons, it is optimal to introduce two generations of products to the market.

For the single generation optimal solution, to determine whether Generation 1 or Generation 2 should be sold, I must compare the relative profit margins to the cost of introduction to market c_{t_m}. For example, under a short planning horizon ($T < \bar{T}$), if c_{t_m} is low and Generation 2 has a high profit margin ($p_2 - c_2$), then it is optimal to sell Generation 2 only, that is introduce now ($t_m^* = 0$). If, again for a short horizon, both generations have equal profit margins ($p_1 = p_2$ and $c_1 = c_2$) and a positive introduction cost ($c_{t_m} > 0$), then I should never introduce Generation 2 ($t_m^* = T$) and sell Generation 1 only.

Furthermore, this threshold is dependent on price, cost, and market characteristics for both of the generations. I find $\bar{T}$ to be decreasing in p_1, c_2, and M_1. A smaller $\bar{T}$ value implies that the optimal solution is more likely to contain two generations of products.

I can also observe that $\lambda_3(0)$ is increasing in these parameters. Thus, as conditions for Generation 1 become more favorable, I am more likely to either never introduce Generation 2 ($t_m^* = T$) for a smaller set of T values or introduce Generation 2 later ($t_m^* > 0$) for a larger set of T values.

The opposite behavior is found for p_2, c_1, and M_2. That is, I have a larger $\bar{T}$ value, which implies that more T values will fall under the threshold. I observe that $\lambda_3(0)$ is decreasing in these parameters. Thus, as conditions for Generation 2 become more favorable, I am more likely to either introduce later ($t_m^* > 0$) for a larger set of T values or never ($t_m^* = T$) for a smaller set of T values.

Notice also that although price is not a part of the sales rate function, it does influence the threshold value and thus the optimal time to market. In the Numerical Analysis section, I also examine the behavior of $\bar{T}$ under various parameter assumptions.

4.3.2.2 A benchmark scenario

For further insights into this problem, I analyze a benchmark scenario. Consider the situation when both generations are equal in price, cost, and market size, and I assume the introduction to market cost is negligible: $p_1 = p_2$, $c_1 = c_2$, $M_1 = M_2$, and $c_{t_m} = 0$. The expression for the threshold value $\bar{T}$ then reduces to the expression in Corollary 4.

Corollary 4: Horizon Threshold for CASE 2 Under a Benchmark Scenario *Under a benchmark scenario with equal generations and zero introduction to market cost,the horizon threshold is derived to be the following:*

$$\bar{T} = \frac{\log(\frac{\psi^2}{\phi^2})}{\phi + \psi} \tag{4–34}$$

Corollary 5: Optimal Time to Market for CASE 2 Under a Benchmark Scenario *Under a benchmark scenario with equal generations and zero introduction to market cost, with a horizon threshold* $\bar{T} = \frac{\log(\frac{\psi^2}{\phi^2})}{\phi+\psi}$*, the optimal time to market can be written as:*

$$t_m^* = \begin{cases} T \text{ (never)}, & T \leq \bar{T}, \\ \frac{T}{2}, & T > \bar{T}. \end{cases} \tag{4–35}$$

Thus, according to Corollary 5, for equal generations, I either sell Generation 1 only (for $T < \bar{T}$) or I bisect the time horizon among the two generations.

I further observe that $\bar{T}/2$ is actually equivalent to the peak sales time $t_p : \frac{\delta f(x_1(t))}{\delta t} = 0$. Then, for values of $T > \bar{T}$, with equal generations, I have $t^*_m = T/2$ which is larger than $\bar{T}/2 = t_p$. This brings us to a conclusion contrary to the current literature.

Corollary 6: Optimal Time to Market vs Peak Sales Under a Benchmark Scenario *Under a benchmark scenario with equal generations and zero introduction to market cost, it is optimal to introduce the second generation after peak sales are realized.*

$$t^*_m = T/2 > t_p \tag{4–36}$$

Where peak sales are defined as $t_p : \frac{\delta f(x_1(t))}{\delta t} = 0$.

Mahajan and Muller (1996) utilize numerical analysis to solve a similar problem with an infinite planning horizon, and find that a "now" or "at maturity" (i.e. when peak sales are reached) rule is optimal. A key difference between my models, however, is that I consider a finite planning horizon. Consequently, my analytic results show that during a finite planning horizon, it is optimal to introduce the second generation "now" or "after maturity" such that each generation is in the marketplace for an equivalent amount of time.

4.3.3 CASE 3: Price and Diffusion Effects

I now consider the most general case for the sales function in which both price and diffusion effect sales. Here, I assume that a_0, a_1, a_2 and b_0, b_1, b_2 are positive constants ($\neq 0$). I then have the following functional forms.

$$f(x_1, p) = a_0 - a_1 p + a_2[\phi(M_1 - x_1) + \frac{\psi}{M_1}(M_1 - x_1)x_1] \tag{4–37}$$

$$g(x_2, p_2) = b_0 - b_1 p_2 + b_2[\phi(M_2 - x_2) + \frac{\psi}{M_2}(M_2 - x_2)x_2] \tag{4–38}$$

Theorem 6: Optimal Prices for CASE 3 *When the sales function includes both price and diffusion effects, the optimal prices p_1^* and p_2^* are:*

$$p_1^* = \frac{\left(\begin{array}{l}(2a_1^2a_2c_1\psi t^2t_m^2 \pm (t_m^2 + a_2t(\phi-\psi)t_m^2 + t^2(-2+a_2(-\phi+\psi)t_m)) \\ \sqrt{a_1^2M_1(4a_2(a_0-a_1c_1)\psi t_m^2 + M_1(4+a_2t_m(4\phi-4\psi+a_2(\phi+\psi)^2t_m)))} \\ +a_1(2a_0a_2\psi t_m^2(-t^2+t_m^2) + M_1(a_2t(\phi-\psi)t_m^2(2+a_2(\phi-\psi)t_m) \\ +t_m^2(2+a_2t_m(\phi-\psi+2a_2\phi\psi t_m)) - t^2(4+a_2t_m(4\phi-4\psi+a_2(\phi^2+\psi^2)t_m)))))\end{array}\right)}{2a_1^2a_2\psi t_m^4} \tag{4–39}$$

$$p_2^* = \frac{\left(\begin{array}{l}b_2M_2(T-t)(T-t_m)^2(\phi^2t+\psi^2t+2\phi\psi T-(\phi+\psi)^2t_m) \\ \pm\frac{-2t^2+T^2+4tt_m-t_m(2T+t_m)}{b_1b_2}[-2b_1M_2+\sqrt{\begin{array}{l}b_1^2M_2(M_2(4+4b_2(\phi-\psi)(T-t_m) \\ +b_2^2(\phi+\psi)^2(T-t_m)^2) \\ +4b_2(b_0-b_1c_2)\psi(T-t_m)^2)\end{array}}\;] \\ \pm\frac{T-t_m}{b_1}[2b_1^2c_2\psi(t-t_m)^2(T-t_m) \\ +(\phi-\psi)(-t+T)\sqrt{\begin{array}{l}b_1^2M_2(M_2(4+4b_2(\phi-\psi)(T-t_m) \\ +b_2^2(\phi+\psi)^2(T-t_m)^2) \\ +4b_2(b_0-b_1c_2)\psi(T-t_m)^2)\end{array}}\;(-t+t_m) \\ +b_1(-2b_0\psi(t-T)(t+T-2t_m)(T-t_m) \\ -M_2(\phi-\psi)(4t^2-T^2+4Tt_m+t_m^2-2t(T+3t_m)))]\end{array}\right)}{2b_1\psi(T-t_m)^4} \tag{4–40}$$

Theorem 7: Optimal Time to Market for CASE 3 *When the sales function includes both price and diffusion effects, the optimal time to market t_m^* is:*

$$t_m^* : \frac{\left(\begin{array}{l} (T-t_m)((-2b_1M_2+b_1b_2M_2(-\phi+\psi)(T-t_m) \\ +\sqrt{\begin{array}{l} b_1^2M_2(M_2(4+4b_2(\phi-\psi)(T-t_m)+b_2^2(\phi+\psi)^2(T-t_m)^2) \\ +4b_2(b_0-b_1c_2)\psi(T-t_m)^2) \end{array}})^2 \\ -\frac{1}{a_1^3a_2^2t_m^4}(a_1M_1(-2+a_2(-\phi+\psi)t_m) \\ +\sqrt{a_1^2M_1(4a_2(a_0-a_1c_1)\psi t_m^2+M_1(4+4a_2(\phi-\psi)t_m+a_2^2(\phi+\psi)^2t_m^2))})^2) \end{array}\right)}{4b_1^3b_2^2\psi^2(T-t_m)^4} \geq c_{t_m} \tag{4–41}$$

I cannot find a closed form solution for t_m^*. Thus, I perform several numerical experiments to further study the behavior of $\lambda_3(t_m)$. I consider a benchmark scenario, as in CASE 2, with equal generations and $c_{t_m} = 0$. As in CASE 2, I find that $t_m^* = T/2$. However, I do not have a threshold behavior for the time horizon in CASE 3. In the Numerical Analysis section, I further analyze the sensitivity of t_m^* to various parameters under the benchmark scenario.

4.4 Numerical Analysis

I perform numerical analysis for CASE 2 and CASE 3. I observe the behavior of the cumulative sales (x_1 and x_2), and sales rates (f and g). I also study the sensitivity of t_m^*, $\bar{T}$, p_1^*, and p_2^* to several parameters. I use the following parameter values from the benchmark scenario discussed in section 4.2.2 in which I consider equal generations. I assume $p > c$ and $0 \leq \phi, \psi \leq 1$; values for ϕ and ψ are values commonly used in empirical literature.

$c_1 = c_2 = 3$ $c_{t_m} = 0$
$M_1 = M_2 = 100$
$\phi = 0.05$ $\psi = 0.5$
$a_0 = b_0 = 0$ $a_1 = b_1 = \begin{cases} 0, \text{CASE 2} \\ 1, \text{CASE 3} \end{cases}$ $a_2 = b_2 = 1$

For CASE 2, I have $a_1 = b_1 = 0$ (no price effect) and $a_2 = b_2 = 1$. For CASE 3, I have $a_1 = b_1 = 1$ and $a_2 = b_2 = 1$.

4.4.0.1 Cumulative sales and sales rate

Let us begin by studying the behavior of the cumulative sales (x_1 and x_2), and sales rates (f and g) for CASE 2 and CASE 3.

In CASE 2, since the optimal price values are the maximum market price, I set $p_1 = p_2 = 5$. For the parameter values defined above, I find $\bar{T} = 8.373$. For now, I will set $T = 10$ (so that $T > \bar{T}$) and $t_m = 5$ ($t_m = T/2$). I use the expressions derived in 4.21, 4.23, 4.29, and 4.30 to create the CASE 2 graphs below. I can observe an increase in cumulative sales and a peak behavior in sales rate for both generations in Figures 4-3 and 4-4, respectively.

I can calculate the peak sales value to be $t_p = \bar{T}/2 = 8.373/2 = 4.187$. Thus I can confirm my theoretical findings from the benchmark case that, for $T > \bar{T}$, the optimal time to market $t_m^* = T/2$ will be greater than peak sales. (For example, for $T = 9 > \bar{T} = 8.373$, I have $t_m^* = T/2 = 4.5 > t_p = 4.187$. Likewise, for $T = 10 > \bar{T} = 8.373$, I have $t_m^* = T/2 = 5 > t_p = 4.187$.)

In CASE 3, for the given parameter values, I calculate $\bar{T} = 89.9$. Thus, I initially set $T = 90$ ($T > \bar{T}$) and find the corresponding time to market to be $t_m = 45$ ($t_m = T/2$).

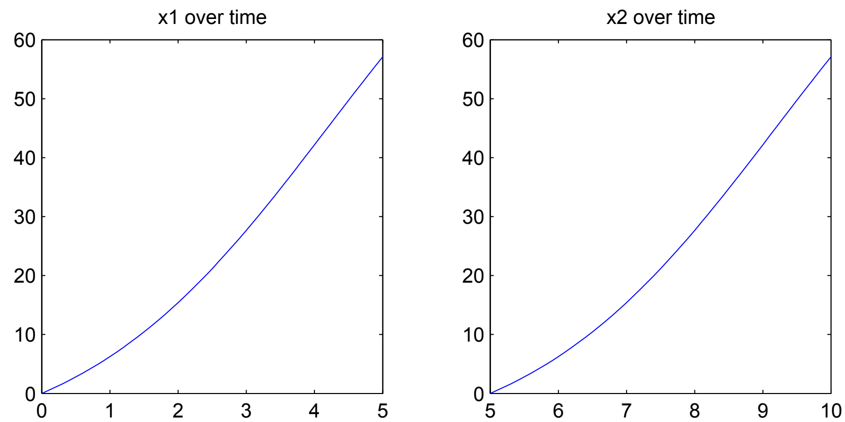

Figure 4-3. CASE 2: Behavior of cumulative sales x_1 and x_2 over time.

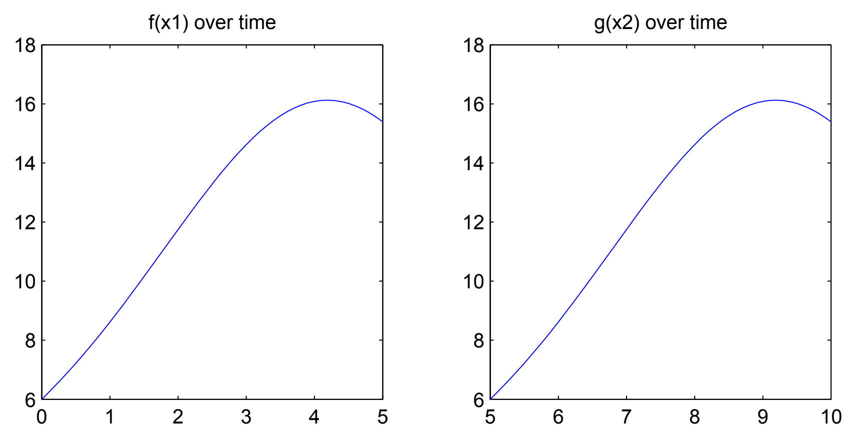

Figure 4-4. CASE 2: Behavior of sales rates $f(x_1)$ and $g(x_2)$ over time.

I use the expressions derived in 4.42, 4.43, 4.52, and 4.59 to create the CASE 3 graphs below. I observe linearly increasing cumulative sales and a constant sales rate for both generations in Figures 4-5 and 4-6, respectively.

Let us discuss further the cumulative sales and sales rate behaviors for CASE 2 and CASE 3. In CASE 2, the cumulative sales are increasing convexly and sales rate is concave. These are the standard behaviors for sales with diffusion effect only and can be found in the diffusion literature.

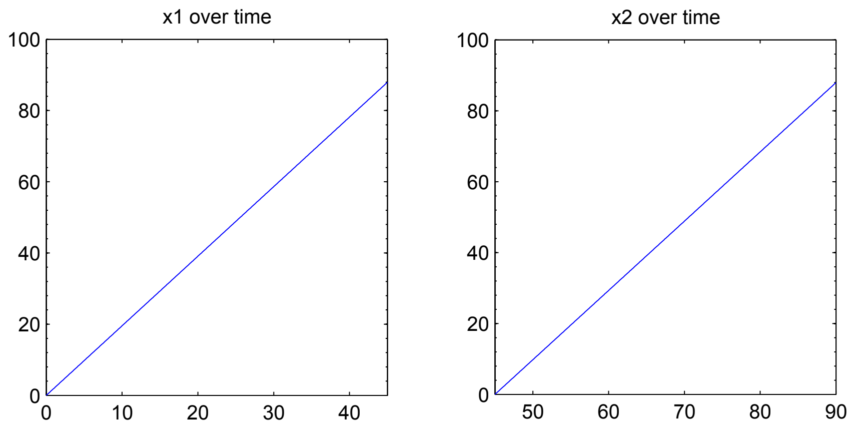

Figure 4-5. CASE 3: Behavior of cumulative sales x_1 and x_2 over time.

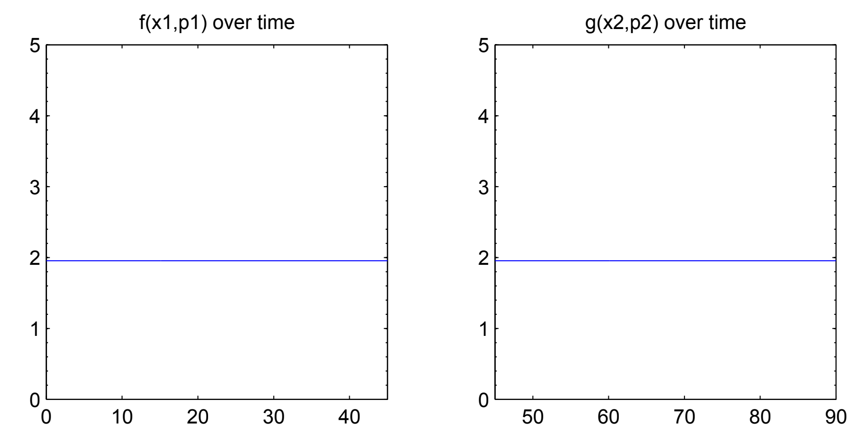

Figure 4-6. CASE 3: Behavior of sales rates $f(x_1, p_1)$ and $g(x_2, p_2)$ over time.

In CASE 3, I have both diffusion and price effect. I find that the cumulative sales becomes linear and the sales rate constant. This is a very different and interesting behavior. It seems that the diffusion effects captured in CASE 2, are offset by the price effect. That is, for cumulative sales, having a positive weight coefficient for price decreases sales, and thus flattens the convex curve found in CASE 2 to a linear curve in CASE 3. For the sales rate, the price coefficient has a negative effect on sales rate which is smaller than the positive effect of the diffusion coefficient, thus yielding a positive constant sales rate. This behavior may be due to my model assumptions. That is, in my sales functions, the price and diffusion effects are additive. These graphs illustrate, that even for the benchmark situation of equal generations, including the effect of price on sales makes a major difference in cumulative sales and sales rate behavior.

I also look at the behavior of p_1^* and p_2^* over time for this benchmark scenario. In Figure 4-7, I see that both p_1^* and p_2^* follow a concave curve. That is, the optimal pricing behavior appears to be initially low so as to increase the sales, and then as the product diffuses to the marketplace it increases, and then decreases again towards the end of the horizon. Since I consider equal generations in this benchmark scenario, p_1^* and p_2^* are identical.

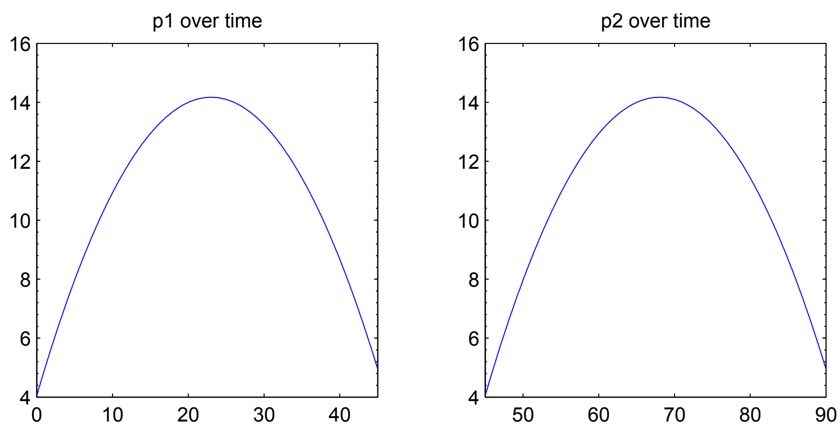

Figure 4-7. CASE 3: Behavior of optimal price p_1^* and p_2^* over time.

4.4.0.2 Sensitivity analysis

I now conduct a sensitivity analysis of my decision variables t_m^*, p_1^*, and p_2^* to the cost, population size, and price and/or diffusion coefficient parameters for CASE 2 and CASE 3. I vary the parameters increasingly according to the following values:

$c_1,\ c_2,\ c_{t_m}$ = (0,5)
$M_1,\ M_2$ = (20, 110)
$a_1,\ b_1$ = (1,3) $a_2,\ b_2$ = (1,1.8)

Increasing parameter	Tthresh	tm*	Profit
c1	increasing	decreasing	decreasing
c2	decreasing	increasing	decreasing
ctm	--	increasing	decreasing
M1	decreasing	increasing	increasing
M2	increasing	decreasing	increasing
a2	decreasing	decreasing	increasing
b2	decreasing	increasing	increasing

Table 4-1. Sensitivity analysis results for CASE 2.

Increasing parameter	tm*	p1(t)	p2(t)	Profit
c1	decreasing	increasing	less variation	decreasing
c2	increasing	less variation	increasing	decreasing
ctm	increasing	less variation	increasing	decreasing
M1	increasing	increasing	more variation	increasing
M2	decreasing	more variation	increasing	increasing
a1	decreasing	decreasing	less variation	decreasing
b1	increasing	less variation	decreasing	decreasing
a2	decreasing	increasing	more variation	increasing
b2	increasing	more variation	increasing	increasing

Table 4-2. Sensitivity analysis results for CASE 3.

The sensitivity results for CASE 2 and CASE 3 are summarized in Tables 6 and 7. For CASE 2, since the horizon threshold $\bar{T}$ is sensitive to the selected parameters and t_m^* depends on $\bar{T}$, I first analyze the sensitivity of $\bar{T}$ and then calculate t_m^* for each parameter value. Similarly, for CASE 3, since p_1^* and p_2^* are functions of t_m, for all parameter changes, I first find t_m^*. (Note: For these experiments, when multiple solutions for t_m^* were found, I evaluated and compared the total profit for each solution and chose the t_m^* which yielded the maximum profit value.)

Let us first examine the results for CASE 2. As c_1 increases, selling Generation 1 becomes less profitable and so t_m^* decreases from a large value close to T (almost "never") to "now" (Generation 2 only). Likewise, as c_2 increases, selling Generation 2 becomes less profitable and so t_m^* increases from "now" (Generation 2 only) to "never" (Generation 1 only). As c_{t_m} increases, it becomes more costly to introduce Generation 2 to the market and so t_m^* increases from $T/2$ to "never" (Generation 1 only). This increase occurs quickly, illustrating how sensitive t_m^* is to c_{t_m}. As M_1 increases, it becomes more profitable to sell Generation 1, thus t_m^* increases from "now" (Generation 2 only) to a high value close to T (almost "never"). Likewise, as M_2 increases, it becomes more profitable to sell Generation 2, thus t_m^* decreases from "never" (Generation 1 only) to a low value close to 0 (almost "now").

To explain the relationship between a_2 and t_m^* (and likewise, b_2 and t_m^*) I must re-consider the definition of a_2 and b_2. These coefficients, which represent the weight of the effect of diffusion on sales, can also be defined as the speed of diffusion. That is, as a_2 increases, the speed of diffusion of sales for Generation 1 increases, meaning that it takes less time to reach maximum sales. Thus t_m^* decreases since Generation 2 can be introduced earlier without losing profit from Generation 1. Likewise, as b_2 increases, the diffusion rate for Generation 2 increases, meaning Generation 2 does not need as much time on the market in order to reach maximum sales, thus t_m^* increases. (Note: t_m^* only increases or decreases slightly from $T/2$. Since the change in t_m^* is small, it shows that these parameters have a minimal effect on t_m^*.).

I also look at the sensitivity of overall profit to these parameters. For CASE 2, profit is decreasing in costs. This is clear for c_1 and c_2 since an increase in these values decreases their corresponding generation's profit margin. For the cost of introducing Generation 2 to the market, since an increase in c_{t_m} increases t_m^* to "never", this means Generation 2 will not be offered, and therefore total profits decrease. Profit is increasing in population size. This is intuitive since a larger population size will increase the sales rate and thus the corresponding generation's profit will also increase, increasing total profit. An increase in the diffusion coefficients a_2 and b_2 also increases profit. Since these coefficients directly effect the sales rate, the corresponding generation's profit component also increases.

For CASE 3, I consider the effect of the parameters on t_m^* and then implicitly on p_1^* and p_2^*. I find results for costs, population size, and diffusion coefficients to be similar to CASE 2. To explain the relationship between a_1 and t_m^* (and likewise between b_1 and t_m^*), I may refer back to the effects of these parameters on the time to market and pricing decisions discussed in the analytical section for CASE 1. I find that t_m^* decreases in a_1 and increases in b_1. As the weight of price increases, the same price will have a larger (negative) effect on sales. That is, an increase in these parameters makes selling the corresponding generation less profitable due to a negative effect on sales and also an implicit decrease in price. Therefore the time each generation is sold on the market decreases. So as a_1 increases, t_m^* decreases, and as b_1 increases, t_m^* increases.

For CASE 3, I also study the sensitivity of the optimal prices p_1^* and p_2^* to these parameters. For costs, I find that as a generation's unit cost increases, its corresponding optimal price also increases. This seems like a natural reaction driven by the maximize profit objective. Thus, as c_1 increases, p_1^* increases, and as c_2 increases, p_2^* increases. When these generation costs increase, the complimentary generation has a decrease in its price variation, or it is more stable over time. For example, as c_1 increases, the variation in p_2^* over time decreases. As c_{t_m} increases, I see that p_2^* increases in order to balance the total margin of offering Generation 2.

As the population sizes increase, the corresponding generation's price increases in order to take advantage of the larger market. For example, as M_2 increases, p_2^* increases. The complimentary generation's price has more variation, that is it is less stable over time. For the same example, if M_2 increases, p_1^* becomes less stable over time. As in CASE 1, as a_1 or b_1 increase, the corresponding generation's price decreases since sales become more sensitive to price. As the diffusion coefficients a_2 and b_2 increase, the corresponding generation's prices increase. This may be motivated by the increased sales from diffusion.

I also look at the sensitivity of overall profit to these parameters. For CASE 3, profit is decreasing in costs. This is clear for c_1 and c_2 since an increase in these values decreases their corresponding generation's profit margin. For the cost of introducing Generation 2 to the market, since an increase in c_{t_m} increases t_m^*, this means Generation 2 will be offered for less time, and therefore total profits decrease. Profit is increasing in population size. This is intuitive since a larger population size will increase the sales rate and thus the corresponding generation's profit will also increase, increasing total profit. An increase in the diffusion coefficients a_2 and b_2 also increases profit. Since these coefficients directly increase the diffusion rate, the corresponding generation's profit component also increases. An increase in the price coefficients a_1 and b_1 decreases profit due to their negative effect on sales.

Overall, these numerical results support my analytical conclusions for t_m^*, and in CASE 3, for p_1^* and p_2^*. The sensitivity of my decision variables to the selected parameters seems fairly intuitive. CASE 3 does offer a more interesting set of results. I can observe the implicit effect of time to market on optimal prices, and the effect on price variation in the complimentary generations. For CASE 2, changes in these parameters may effect whether one or two generations will be sold. For CASE 3, usually two generations are always optimal, although the time to market may change significantly. A manager may consider the sensitivity of time to market and pricing when choosing or negotiating his unit costs, deciding what population size to market to, or in studying the weight of price and diffusion in the market.

Please refer to Chapter 6 for the related conclusions and future research extensions.

CHAPTER 5
OPTIMAL NUMBER OF GENERATIONS FOR A MULTI-GENERATION PRICING AND TIMING MODEL IN NEW PRODUCT DEVELOPMENT

5.1 Introduction

When examining a new product development (NPD) marketing scenario, there are several operations management (OM) and marketing decisions to be considered. Operations decisions may include production timing and quantity, capacity investments and constraints, and time-to-market decisions. The marketing perspective may consider pricing decisions, and investments in quality and innovation speed. The NPD literature has examined these decisions in a primarily disjoint fashion. This chapter contributes to the OM/Marketing Interface literature by solving both pricing and time-to-market decisions simultaneously. In addition, there has been limited literature discussing the optimal innovation speed, or clockspeed, for NPD products. In this chapter, I solve for the optimal number of generations of an NPD product, when sales are a function of both pricing and diffusion.

The remainder of this chapter is organized as follows. In Section 2, I describe in detail my model assumptions and mathematical objective. In Section 3, I perform my analysis using optimal control theory to derive expressions for the optimal number of generations and optimal price per generation. In Section 4, I perform numerical experiments to examine the behavior of these optimal expressions as well as a sensitivity analysis to all parameters.

Please refer to the Chapter 2 for a review of the related literature.

5.2 Model

I consider multiple generations of a new product sold over a finite time horizon, T. Each product of generation i is produced at a unit cost $c_i(t)$ and sold at a price $p_i(t)$ to a population of size M_i. Each new generation is introduced to the market at a fixed cost $c_{t_{m_i}}$. I consider a single rollover scenario, in which sales of Generation i will stop once Generation $i+1$ is introduced to the market.

Our objective is to determine the optimal price for each generation, $p_i^*(t)$, the optimal introduction time for each generation, $t_{m_i}^*$, and the optimal number of generations, n^*, such that total profit is maximized over the time horizon. The objective function is stated mathematically in Equation 3.1.

$$\Pi = \sum_{i=1}^{n} \int_0^{t_{m_i}} \dot{x}_i(t)(p_i(t) - c_i(t))dt - \sum_{i=1}^{n} c_{t_{m_i}} v_{t_{m_i}} \tag{5–1}$$

Where I have sales rate defined as:

$$\dot{x}_i = a_{0_i} - a_{1_i} p_i + a_{2_i}(\phi(M_i - x_i) + \frac{\psi}{M_i}(M_i - x_i)x_i) \tag{5–2}$$

I define total profit as the net revenue for each generation earned over the time horizon, minus the cost of introduction to market for each generation. The objective function may be considered as an extension of the two-generation pricing and timing model from Chapter 4. Since I assume a finite time horizon T, I do not include discounting.

The sales rate is an additive model of both price and diffusion effects on sales. The diffusion component is based on the Bass (1969) [10] diffusion model. I have a_{0_i}, a_{1_i}, and a_{2_i} as coefficients for initial sales, pricing effect, and diffusion effect, respectively, for each generation i.

The variable $v_{t_{m_i}}$ is an impulse variable which is used to determine whether or not Generation i is introduced to the market. If it is introduced, the corresponding fixed cost $c_{t_{m_i}}$ is incurred.

To simplify this non-linear optimization problem, let us assume all generations to be equal. Thus I have: $p_i = p$, $c_i = c$, $c_{t_{m_i}} = c_{t_m}$, $\dot{x}_i = \dot{x}$ which implies $M_i = M$ and $x_i = x$. I then have the following profit expression:

$$\Pi = n \int_0^{t_m} \dot{x}(t)(p(t) - c(t))dt - nc_{t_m} \tag{5–3}$$

Notice that the impulse variable $v_{t_{m_i}}$ has been transformed implicitly in the number of generations to be sold, n. Our decision variables are now $p(t)$, t_m, and n.

Based on results from the benchmark scenario of both CASE 2 and CASE 3 from Chapter 4, I assume that all generations (of equal characteristics) will be spaced out equally in the time horizon. That is, I assume $t_m = T/n$. This assumption is also made in Carrillo (2004) [22]. Morgan, Morgan and Moore (2001) [21] note that equal spacing of generations may not be optimal, but still yields good solutions. Carrillo (2005) [23] points out that the empirical literature supports pacing new products at regular time intervals.

I would like to note that my assumptions are made based for a top-level strategic perspective. I am considering an aggregate planning model to determine an optimal time-pacing strategy. Thus, assuming equal generations paced at equal intervals provides insight into this top-level strategy for both the optimal number of generations and the optimal pricing per generation. These assumptions also allow us to derive closed form expression for my optimal decisions.

Thus, I have the following model:

$$\text{Max} \qquad \Pi = n \int_0^{T/n} \dot{x}(t)(p(t) - c(t))dt - nc_{t_m} \tag{5–4}$$

$$\text{s.t.} \qquad \lambda(t) : \dot{x}(t) = f(x(t), p(t)) \tag{5–5}$$

Where the sales rate function is:

$$f(p(t), x(t)) = a_0 - a_1 p(t) + a_2(\phi(M - x(t)) + \frac{\psi}{M}(M - x(t))x(t)) \tag{5–6}$$

A summary of my notation is given below.

n	number of generations to introduce
t_m	time at which a new generation is introduced to the market
$p(t)$	dynamic price offered per generation
T	length of the planning horizon
$c(t)$	unit cost at time t of the current generation
c_{t_m}	fixed cost of introducing the next generation to the market at time t_m
$\dot{x}(t)$	sales rate at time t of the current generation
$x(t)$	cumulative sales in the time interval $(0, t)$ of the current generation
$\lambda(t)$	marginal value of selling one unit of the current generation
a_0	positive constant for initial sales
a_1	positive constant for price effect on sales rate
a_2	positive constant for diffusion effect on sales rate
M	market size of each generation
ϕ	coefficient of innovation
ψ	coefficient of imitation

5.3 Analysis

I solve my model using Optimal Control Theory, since I have the dynamic nonlinear optimization for $p(t)$. Optimal Control Theory is a non-linear optimization methodology utilized for dynamic economic problems. For a summary of the methodology and applications, refer to Sethi and Thompson (2000) [15]. For my model, the control variable is the price for each generation, $p(t)$. The state variable is the cumulative sales for each generation, $x(t)$. The adjoint variable is the marginal value defined in the problem constraint: $\lambda(t)$. Similar to Gaimon and Morton (2005) [35], I will solve for the optimal number of generations, n^*, using first order conditions (f.o.c.) of the profit expression. I define the Hamiltonian as follows. (Please note: From this point forward, I remove the time variable for clarity.)

$$H = (\lambda + n(p - c))(a_0 - a_1 p + a_2(\phi(M - x) + \frac{\psi}{M}(M - x)x))) \tag{5–7}$$

To find the optimal price $p^*(t)$, I first use the f.o.c. typically associated with Optimal Control Theory to derive an expression for p^* in terms of cumulative sales x and the marginal value of sales λ. I then solve for x and λ using simultaneous differential equations and Optimal Control methods. I then substitute back in the derived x^* and λ^* expressions to find the expression for p^*.

Theorem 1: Optimal Price per Generation *For equal generations and sales rate as a function of both price and diffusion, the optimal price per generation $p^*(t)$ is:*

$$p^*(t) = \frac{\left(\begin{array}{l} 2a_1^2 a_2 c\psi t^2 \frac{T^2}{n} - (\frac{T^2}{n^2} + a_2(\phi - \psi)t\frac{T^2}{n^2} + t^2(-2 + a_2(-\phi + \psi)\frac{T}{n})) \\ \sqrt{\left(a_1^2 M n^2 (4a_2(a_0 - a_1 c)\psi \frac{T^2}{n^2} + M(4 + a_2\frac{T}{n}(4\phi - 4\psi + a_2(\phi + \psi)^2\frac{T}{n})))\right)} \\ +a_1 n(2a_0 a_2 \psi \frac{T^2}{n^2}(-t^2 + \frac{T^2}{n^2}) + M(a_2(\phi - \psi)t\frac{T^2}{n^2}(2 + a_2(\phi - \psi)\frac{T}{n}) \\ +\frac{T^2}{n^2}(2 + a_2\frac{T}{n}(\phi - \psi + 2a_2\phi\psi\frac{T}{n})) - t^2(4 + a_2\frac{T}{n}(4\phi - 4\psi + a_2(\phi^2 + \psi^2)\frac{T}{n})))) \end{array}\right)}{2a_1^2 a_2 \psi \frac{T^4}{n^3}} \tag{5–8}$$

Here I have assumed equal generations offered for equal time, and thus $t_m = T/n$. I must note that a_1, a_2, and ψ should never be zero. [Please refer to Appendix C for proofs of all theorems.]

I am also able to show that the optimal price $p^*(t)$ is concave over time for constant costs and relatively large initial sales. This result is similar to those shown in Kalish (1983) [12] for a single generation of a new product.

Corollary 1: Optimal Price Concave over Time *For equal generations and sales rate as a function of both price and diffusion, assuming constant costs $c(t) = c$ and initial sales relatively larger than cost such that $a_0 - a_1 c \geq 0$, the optimal price per generation $p^*(t)$ is concave over time.*

Using $p^*(t)$ as derived above, I can restate my maximum profit objective function as follows.

$$\Pi(p^*(t)) \quad = \quad n\int_0^{T/n} \dot{x}(p^*(t))(p^*(t)-c(t))dt - nc_{t_m} \tag{5–9}$$

Again, assuming equal generations offered for equal time to market for each generation, I have $t_m = T/n$. Making this substitution in Π, when s.o.c. for concavity of profit are met, I can find n^* from the f.o.c. to be the following.

Theorem 2: Optimal Number of Generations *For equal generations and sales rate as a function of both price and diffusion, and assuming that each generation is offered for an equal amount of time ($t_m = T/n$), when the second order conditions for profit hold, the optimal number of generations to offer, n^*, is the following:*

$$n^* = \frac{\begin{array}{l}((8(3^{(2/3)})a_0^2a_1a_2c_{t_m}M^2(\phi-\psi)+8(3^{(2/3)})a_1^3a_2cc_{t_m}M(cM(\phi-\psi)+6c_{t_m}\psi)\\ -16(3^{(2/3)})a_0a_1a_2c_{t_m}M(a_2M^2\phi(\psi-\phi)+a_1(cM(\phi-\psi)+3c_{t_m}\psi))\\ +3^{(2/3)}a_1^2a_2^2c_{t_m}M^2(16cM\phi(\psi-\phi)+3c_{t_m}(\phi^2-18\phi\psi+\psi^2))\\ +3^{(1/3)}K^{(2/3)}+a_1a_2c_{t_m}M(\phi-\psi)(8(3^{(2/3)})a_2^2M^3\phi^2-9K^{(1/3)}))T)\end{array}}{24a_1c_{t_m}MK^{(1/3)}} \tag{5–10}$$

Where I define K as:

$$K = \begin{array}{l}(a_1^2a_2c_{t_m}^2M(-192a_1^3c^3M\psi+2\sqrt{6}\\ \sqrt{\left(\begin{array}{l}\frac{-1}{a_1c_{t_m}}((M(a_0-a_1c+a_2M\phi)(8(a_0-a_1c)^2\\ -a_2(9a_1c_{t_m}-16a_0M+16a_1cM)\phi\\ +8a_2^2M^2\phi^2)+9a_1a_2c_{t_m}(a_1(2c_{t_m}-cM)\\ +M(a_0+a_2M\phi))\psi)^2(a_2M^2(\phi-\psi)^3-24a_1c_{t_m}\psi^2))\end{array}\right)}\\ -24M(a_0+a_2M\phi)^2(-8a_0\psi+a_2M(3\phi^2-14\phi\psi+3\psi^2))\\ -72a_1^2(-8a_0c^2M\psi+a_2(3cc_{t_m}M(\phi-\psi)\psi+6c_{t_m}^2\psi^2+c^2M^2(\phi^2-10\phi\psi+\psi^2)))\\ +9a_1M(-64a_0^2c\psi+8a_0a_2(3c_{t_m}(\phi-\psi)\psi+2cM(\phi^2-10\phi\psi+\psi^2))\\ +a_2^2M(16cM\phi(\phi^2-6\phi\psi+\psi^2)+c_{t_m}(\phi-\psi)(\phi^2+22\phi\psi+\psi^2)))))\end{array} \tag{5–11}$$

Note that this result will only be optimal if the s.o.c. of profit hold; that is, I must show that profit is concave in n in order to use f.o.c. to find the optimal n^* which maximizes the profit function Π. I am not able to show that this s.o.c. (specifically $\frac{\delta^2\Pi}{\delta n^2} \leq 0$) always holds nor derive analytical conditions under which it holds. However, for specific numerical instances, I can easily check this s.o.c. to ensure that the n^* value found from the f.o.c. is indeed optimal. Please refer to the Appendix for a complete expression of this s.o.c. for profit to be concave in n.

This analytical expression allows any manager who can set reasonable parameter values to determine the optimal number of generations to offer in a given time horizon. The optimal number of generations, n^*, will also determine the optimal amount of time each generation should be on the market: $t_m^* = T/n^*$.

5.4 Numerical Experiments

I now perform numerical experiments to draw further insights from my analytical results. I consider a benchmark scenario of equal generations with the following parameter values.

$T = 100$
$c = 3$
$c_{t_m} = 50$
$a_0 = 25$
$a_1 = 1$
$a_2 = 1$
$M = 100$
$\phi = 0.05$
$\psi = 0.5$

Note that I assume the fixed cost per generation, c_{t_m}, is much higher than the unit cost, c. I would also like to explain that based on the other parameter values, the parameter a_0 must have a relatively high value in order to have positive sales values over time for the price component of the sales rate ($a_0 - a_1 p(t) \geq 0$, $\forall t$).

Using these parameter values, I found the optimal number of generations to be $n^* = 24.015$, yielding an optimal time to market $t_m^* = T/n^* = 4.164$. Thus, under these parameter values for equal generations, a manager would sell 25 generations for 4.164 time units each. The profit achieved with this optimal strategy is \$28,679.

Profit as a function of n is graphed in Figure 5-1. I am able to check the s.o.c. of profit under these parameter values and find profit to be concave in n for $n \leq 41.25$. Thus, my n^* value of 24.015 found from the f.o.c. result is indeed optimal.

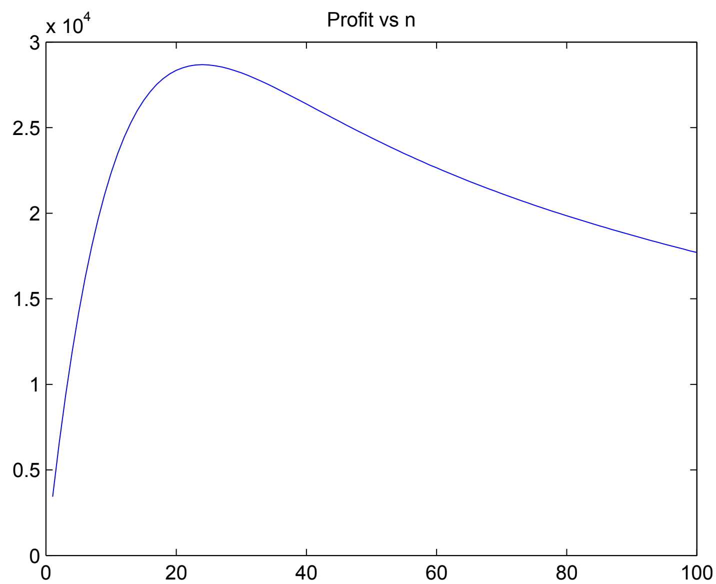

Figure 5-1. Profit as a function of n.

The optimal price over time is shown in Figure 5-2. The optimal price at time $t = 0$ is $p^*(0) = \$13.07$. Price then increases to a maximum value of $\hat{p}^*(t) = \$23.19$, and decreases to $p^*(t_m) = \$20.21$ before the next generation is introduced. This concave increasing/decreasing behavior of optimal price reflects an initial attempt to attract customers followed by an increase in price once a solid market position is attained finishing with a decreasing price as the diffusion of sales declines.

I can better understand the behavior of the optimal price $p^*(t)$ by examining the behavior of the sales rate $f(p(t), x(t))$ over time. Recall that from Equation 3.2 that the

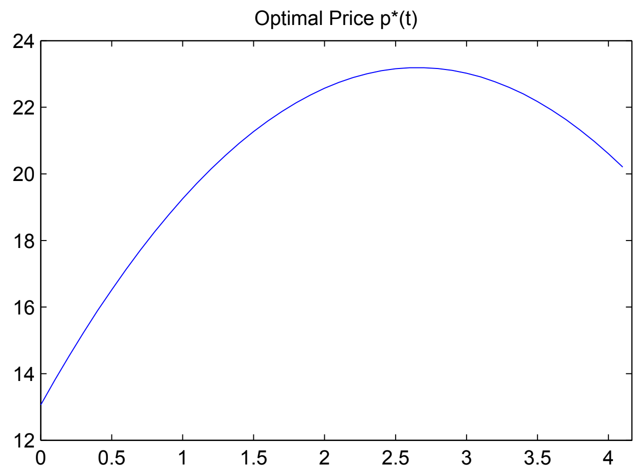

Figure 5-2. Optimal Price $p^*(t)$, over time.

sales rate $f(p, x)$ is comprised of a negative price component and a positive diffusion component. I graph these two components separately in Figure 5-3.

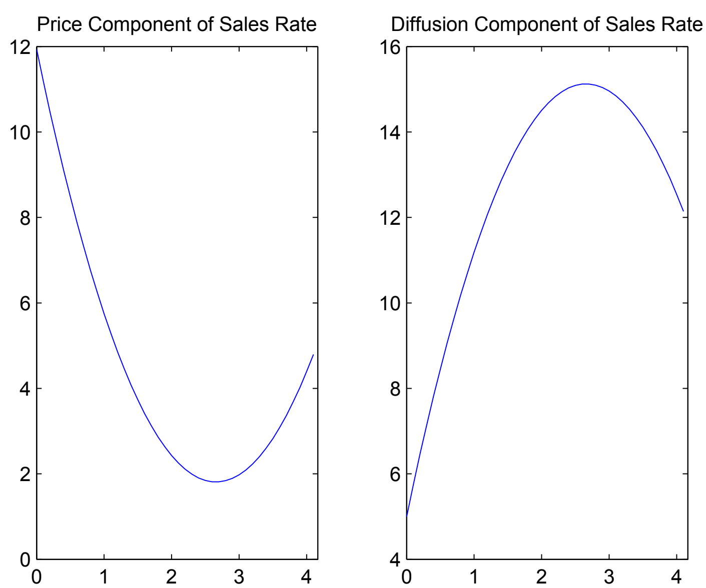

Figure 5-3. Price Component and Diffusion Component of Sales Rate, over time.

Over time, the diffusion component of the sales rate is increasing then decreasing. This is comparable to the literature on NPD diffusion and the Bass Model (see for example Bass (1969) [10] and Norton and Bass (1987) [11]). I believe that this diffusion behavior of sales affects the optimal price, such that it is also increasing then decreasing. I observe that the time at which the maximum optimal price is offered, $\hat{t}$, is equal to the time at which peak sales from the diffusion component occur, $t_p = 2.7$. Since I have a negative price effect on sales plus a positive diffusion effect, my sales rate becomes constant at $f(p, x) = 16.93$ (see Figure 5-4).

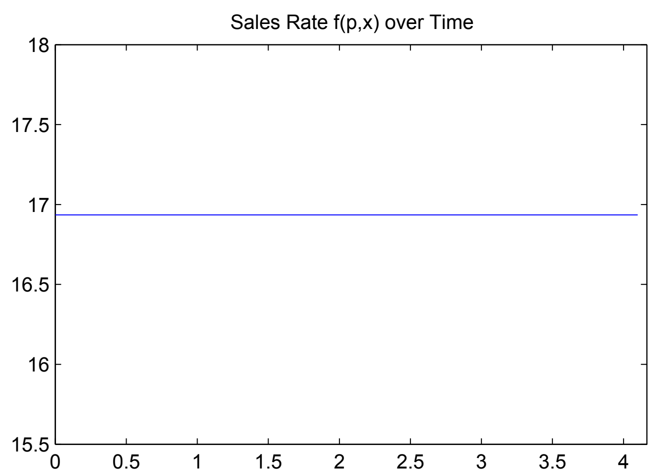

Figure 5-4. Sales Rate $f(p, x)$, over time.

I perform a sensitivity analysis on several parameters affecting the optimal number of generations (and thus also the optimal time to market), the optimal price over time and maximum price offered, and the total profit. I vary the following parameter values over the range specified below:

$T = (10,\ 150)$
$c_{t_m} = (1{,}150)$ $c = (0{,}10)$
$M = (10,\ 150)$
$a_0 = (22,\ 40)$ $a_1 = (1{,}6)$ $a_2 = (1{,}6)$

Increasing parameter	n*	tm*	Profit	p*(t)	p*max
T	increasing	constant	increasing	same	constant
ctm	decreasing	increasing	decreasing	increasing	increasing
c	decreasing	increasing	decreasing	increasing	increasing
M	decreasing	increasing	increasing	more variation	increasing
a0	increasing	decreasing	increasing	increasing	increasing
a1	decreasing	increasing	decreasing	decreasing	decreasing
a2	increasing	decreasing	increasing	more variation	increasing

Table 5-1. Summary of Sensitivity Analysis Results.

A summary of my results is shown in the table below.

I will discuss these results one parameter at a time. For each parameter, I will discuss the sensitivity of the decision variable for the changes in that parameter value. Referring to Table 5-1, I will examine one row at a time and discuss the results shown in the corresponding column variables. As a quick note, among the variables that I examine (columns in Table 5-1), the price $p(t)$ is the only dynamic variable. I summarize the overall change in behavior for price over time. I explain in more details below what each of these descriptions imply.

Let us first discuss the parameter T, the time horizon. As the time horizon increases, the optimal number of generations to offer, n^*, increases. That is, the more time I have to sell, the more generations I will offer. Since I assume $t_m = T/n$, and since n is increasing linearly in T, I find t_m to remain constant as T increases. That is, the time I offer each generation is not affected by the length of the horizon since I am increasing the number of generations offered.

I find profit increasing in T; this is expected since as the number of generations increases so do my total sales. I find the optimal price over time to be unaffected by changes in T, and thus the maximum price is constant.

For increasing values of c_{t_m}, I find n^* to be decreasing. That is, the more costly it is to introduce a new generation, the fewer number of generations will be introduced.

Since I have an inverse relationship between n^* and t_m^* ($t_m^* = n^*/T$), I thus find t_m^* to be increasing. That is, if I offer fewer number of generations over the time horizon, then each generation will be offered for more time. As development costs increase, profit is decreasing.

As c_{t_m} increases, I find an increasing shift in the optimal price. That is, the concave behavior of price still holds, but the entire price curve shifts upward as costs increase. This creates a larger maximum price to counter the higher costs. I illustrate this affect on price in Figure 5-5. Notice the overall increase in the price curve from the solid line to the dashed line. Although price is increasing in development costs, it does not seem to be very sensitive; that is, over a large increase in c_{t_m} there is a slight increase in $p(t)$ over time.

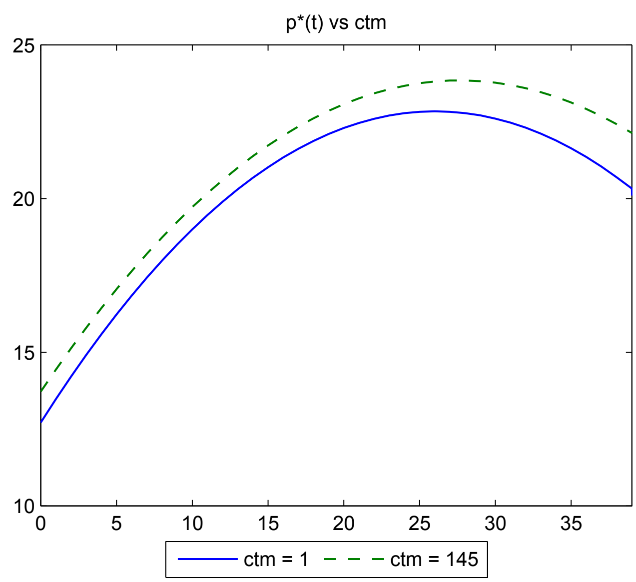

Figure 5-5. Increasing Shift in Price Curve: $p^*(t)$ vs. c_{t_m}.

Similarly, as unit costs, c, increase, the number of generations offered, n^*, decreases and thus the time each generation is offered, t_m^*, increases. Intuitively, as costs increase, profit decreases. In response to increasing c, I find that prices also increase throughout tm; that is, the price curve shifts up. The maximum price thus also increases.

For market size, M, the larger my market, the fewer number of generations I should offer. This implies that t_m^* is increasing in M while n^* is decreasing. This result reflects on the impact of diffusion on sales. That is, the larger my market size, the more time I should allow my product to diffuse into the market. Thus, I find a higher t_m^* value. Even though the number of generations decreases in market size, my total profit increases from the positive effect of increased t_m on sales. I find price to have more variation over time as M increases. That is, the peak in the price curve is steeper with larger values of M, meaning that for larger population size the change in price over time is more dramatic. This larger variation in prices may be possible since a larger market may increase the probability of sales. This may also explain the increase in the maximum price offered.

Lastly, I examine the sales rate coefficient parameters. For a_0, the initial sales, an increasing value leads to more generations and a decreasing time on the market for each generation. The profit is increasing, as is price. For a_1, the customer's sensitivity to price, an increase leads to fewer generations being offered for more time each. That is, as customers become more sensitive to price they are less willing to buy at the same price and thus fewer products are offered. Both profit and price decrease as a_1 increases. For a_2, the speed of diffusion, an increase yields a larger number of generations offered for less time each. That is, as diffusion speed increases, the amount of time necessary to gain sales is less, and thus t_m^* decreases, driving up n^*. Since the number of generations increases, and thus total sales, profit is also increasing in a_2. As a_2 increases, I find more variation in the optimal price. That is, the change in price over time is more noticeable. This creates a larger maximum price. I illustrate this affect on price in Figure 5-6. As the price curve changes from the solid line to the dashed line, variation in the price value increases. (In Figure 5-6, note that the price curve for $a_2 = 6$ actually ends at $t = 15$, which is the corresponding length of time on the market, t_m, for this a_2 value.)

Overall, these numerical results support my analytical conclusions. The sensitivity of my decision variables to the selected parameters seems fairly intuitive. I can observe the

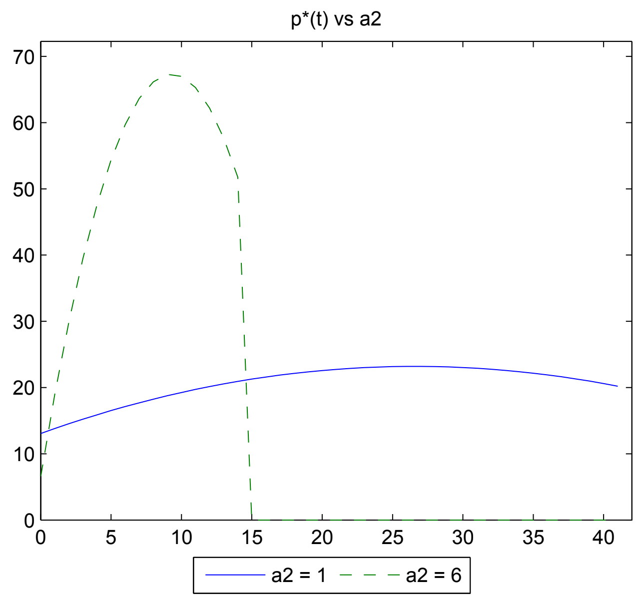

Figure 5-6. Increase in Price Variation: $p^*(t)$ vs. a_2.

effect of the optimal number of generations, or likewise the optimal time to market, on the optimal price over time, variation in price, and maximum price offered. A manager may consider these reported sensitivities when determining his time horizon, negotiating his unit costs, deciding what population size to market to, or in studying the weight of price and diffusion in the market.

Please refer to Chapter 6 for the related conclusions and future research extensions.

CHAPTER 6
CONCLUSIONS

In this chapter, I review the conclusions of each of the research chapters and highlight how the results of each research contribute to the related literature. I also discuss future research extensions for each of these research topics as well as for the general OM/Marketing Interface research, which is the focus of this dissertation.

6.1 Inventory Management under Advance Selling: Optimal Order and Pricing Policies

In this paper, I consider inventory order and pricing decisions for an advance selling marketing strategy. In the advance selling strategy, there are two periods: the advance sales period, succeeded by the spot sales period. I assume that consumption occurs at the end of the spot period. I assume that an order quantity Q is placed at the beginning of the advance sales period and some portion of this inventory X_a is allocated a priori for advance sales. I announce Q, X_a, and the prices p_a and p_s before the advance sales period.

Customers decide whether or not to purchase based on their valuation V of the product. I assume this valuation is not realized until the beginning of the spot period. In the advance sales period, customers compare their expected utility of advance purchasing versus waiting to spot purchase. These expected utility expressions include the expected future valuation and the probability of finding inventory (β). I derive a maximum advance sales price $\hat{p}_a$ for which the expected utility of an advance purchase is greater than or equal to the expected utility of waiting to purchase in the spot period. I find that if I set my advance sales price to this maximum price, I can induce all advance sales customers to purchase. I thus assume that upon deciding the advance sales inventory level, X_a, I will advance sell to $N_a = X_a$ customers, of whom all will advance purchase at the price $p_a = \hat{p}_a$.

I thus seek to determine the optimal order policy (Q^*, X_a^*) and optimal pricing policy (p_a^*, p_s^*) which maximized total expected profit $E[\Pi]$ attained by both advance sales and spot sales.

Bernoulli Customer Valuations			
Sensitivity Analysis	**Q(H)**	**Xa(H)**	**E[Π(H)]**
α INC	CONST (*Xa_H > 0*) then INC (*Xa_H = 0*)	DEC then = 0	INC
H-L Spread DEC	DEC (*Xa_H > 0*) then CONST (*Xa_H = 0*)	0 then INC	DEC

Uniform Customer Valuations			
Sensitivity Analysis	**Q(H)**	**Xa(H)**	**E[Π(H)]**
p_s DEC	DEC (*Xa_H > 0*) then INC (*Xa_H = 0*)	DEC then = 0	DEC
H-L Spread DEC with $F_v(p_s)$ DEC	DEC	DEC	DEC

Table 6-1. Summary of Sensitivity Analysis Trials

Assuming a Bernoulli customer valuation distribution, I find the optimal order quantity Q^* for a given X_a for both the case when $p_s = H$ and $p_s = L$. When $p_s = L$, $Q^* = M$ under a given condition and $Q^* = 0$, otherwise. When $p_s = H$, Q^* has a standard newsvendor component plus the advance sales inventory X_a.

I then find the optimal advance sales inventory level X_a^* for a given Q for both p_s cases. In both cases, I find an extreme point solution for X_a. When $p_s = L$, I find X_a^* equal to its upper bound Q under a given condition and $X_a^* = 0$, otherwise. When $p_s = H$, I find X_a^* equal to its upper bound $M - \frac{\alpha k^2}{(1-\alpha)} < Q$ under a given condition and $X_a^* = 0$, otherwise.

I find the optimal order policy (Q^*, X_a^*) by solving for X_a^* after substituting $Q = Q^*$ into the expected profit expression. I find the optimal policy when $p_s = L$ to be $(Q^*, X_a^*) = (M, 0)$; that is I never advance sell when the spot price is low. For $p_s = H$, I find $(Q^*, X_a^*) = (M, M - \frac{\alpha k^2}{(1-\alpha)})$ (advance sell to almost everyone) under a given condition and $(Q^*, X_a^*) = (F_{D_s}^{-1}(\frac{H-c}{H}), 0)$ otherwise.

I determine the optimal pricing policy to be $(p_a^*, p_s^*) = (L + (H - L)\alpha, H)$ for a given condition and $(p_a^*, p_s^*) = (L, L)$ otherwise.

I perform several numerical experiments to analyze the sensitivity of my analytical results to the customer valuation parameters α, H, and L, and to better understand the behavior of expected profit in the advance sales inventory decision, X_a. I perform a sensitivity analysis on the effect of the valuation probability α and the spread between the high and low valuation levels ($H - L$ spread) on the optimal values Q^*, X_a^*, and the expected profit. I also perform experiments for a Uniform customer valuation distribution. A summary of all of these experiments is shown in Table 6-1.

My contributions to the literature include analyzing the advance sales inventory decision X_a as well as the total order decision Q. I show an extreme point solution for the advance sales inventory X_a leading to a "go/no-go" advance sales decisions. I also perform extensive numerical experiments on the sensitivity analysis of the customer valuation parameters α, H, and L. I find a threshold behavior in the α and $H - L$ Spread values in determining whether or not to advance sell. I also extend my study to consider a different customer valuation distribution.

I believe further research can be done to examine other customer valuation distributions. Future research may also consider additional inventory cost parameters, such as goodwill penalty and salvage value.

6.2 Multi-Generation Pricing and Timing Decisions in New Product Development

In my research, I study the optimal time to market and pricing decisions of a two generation product. I consider three cases for my sales function: CASE 1, price effect only; CASE 2, diffusion effect only; and CASE 3, price and diffusion effect.

I develop analytical expressions for the optimal decisions in all three cases. For CASE 1, I find a "now" or "never" result for time to market. The optimal time to market is also a function of the optimal prices which are linear in costs. For CASE 2, I derive a threshold value for the time horizon which determines whether Generation 2 will be offered sometime within the horizon or never. I also find that when two generations are sold, the optimal introduction time for Generation 2 occurs after peak sales of Generation 1. For CASE 3, I derive optimal price expressions and find that a two-generation scenario seems to be optimal almost all of the time.

I perform numerical analysis for an equal generation benchmark case, which supports my analytical results. I examine the sensitivity of my decision variables to cost parameters, population size, and price and diffusion weight parameters for CASE 2 and CASE 3.

I am considering some extensions to this research. One such extension would be to modify my sales function such that price and diffusion effects are not simply additive. This should result in a sales rate that is not constant, as I found in CASE 3. To adjust my model, I may consider replacing the diffusion coefficient, a_2 and b_2, with some function of price: $A_2(p_1)$ and $B_2(p_2)$. Another extension would be to consider the problem of determining the optimal number of generations to introduce in a given time horizon under my joint pricing and timing decision model.

6.3 Optimal Number of Generations for a Multi-Generation Pricing and Timing Model in New Product Development

In Chapter 5, I look at strategic decisions for a multi-generation new technology product with sales as a function of both price and diffusion. I derive an analytical expression for the optimal dynamic price $p^*(t)$ and also show that price is concave over time. I also derive an analytical expression for the optimal number of generations n^* given that the second order conditions hold. I assume equal generations and each generation to be offered for an equal amount of time, thus from my result for n^*, I am able to determine $t_m^* = T/n^*$. My analytics employ optimal control theory. An extensive numerical analysis is also performed.

I believe my analytical results as well as my numerical experiments contribute to the new product development (NPD) literature and Operations Management (OM) / Marketing Interface literature by highlighting the decision of the optimal number of generations to introduce. In comparison with innovation speed, or clockspeed, papers, I consider a specific additive model of sales with both pricing and diffusion effects and simultaneously solve for optimal pricing, timing, and number of generations for a maximum profit objective.

As an extension, I may consider the clockspeed n as a factor in my sales rate function. This is based on some of the literature (mostly empirical) which has commented that an increased clockspeed can create a positive perception of the product on the market. I may have a sales rate increasing in clockspeed up until some threshold and then decreasing beyond some number of generations. Consider the following sales rate:

$$\dot{x} = a_0 - a_1 p + a_2(\phi(M - x) + \frac{\psi}{M}(M - x)x) + S(n) \tag{6–1}$$

Where $S(n)$ is some function of n. I may also consider replacing the constant a_2 with a function $A_2(p)$ dependent on price. This may replace the additive contribution of price to sales with a function in which the diffusion effect includes a multiplicative price effect.

My model may also be re-examined to consider the scenario when generations are not offered for equal time, thus solving for n^* and t_m^* separately. The model can also be expanded to consider unique generations and the optimal values for $t_{m_i}^*$, and $p_i^*(t)$ for each generation i.

One could also consider a capacity constraint on sales. That is, the sales rate $\dot{x}(t) \leq Z$, where Z is some fixed capacity. Another extension would be to examine the model under an infinite time horizon $T = \infty$. This would require adding discounting to the profit objective.

6.4 OM/Marketing Interface

The OM/Marketing Interface research area is new and quickly expanding research area. There has been substantial evidence for the need to consider multi-disciplinary perspectives when making supply chain decisions. The goal of this dissertation has been to examine various research questions which combine both operations management and marketing decisions. The OM questions I consider include inventory management and NPD timing and production decisions. I include the marketing decision of pricing in each of these OM scenarios. I feel that my work has made a contribution to the growing OM/Marketing Interface research literature, and I hope that my results can be implemented by supply chain managers.

APPENDIX A
PROOFS FOR CHAPTER 3: INVENTORY MANAGEMENT UNDER ADVANCE SELLING: OPTIMAL ORDER AND PRICING POLICIES

Proof of Theorem 1: Optimal Q^* for Bernoulli Customer Valuations with $p_s = L$ I derive the optimal inventory Q^* as follows.

$$\begin{aligned} E[\Pi(L)] &= (L + (H-L)\alpha(\frac{M-Q}{M-X_a}) - c_a)X_a \\ &+ L(Q - X_a) - cQ \end{aligned} \tag{A–1}$$

s.o.c.

$$\frac{\delta E[\Pi(L)]}{\delta Q]} = \frac{-X_a(H-L)\alpha}{M-X_a} + L - c \tag{A–2}$$

$$\frac{\delta^2 E[\Pi(L)]}{\delta Q^2]} = 0 \tag{A–3}$$

Proof of Theorem 2: Optimal Q^* for Bernoulli Customer Valuations with $p_s = H$

I approximate the spot demand with a Normal distribution with mean $\mu_{D_s} = (M - X_a)\alpha$ and standard deviation $\sigma_{D_s} = \sqrt{(M - X_a)\alpha(1-\alpha)}$. I derive the optimal inventory Q^* as follows.

$$\begin{aligned} E[\Pi(H)] &= (L+(H-L)\alpha-c_a)X_a \\ &\quad +H((M-X_a)\alpha-\Lambda_{D_s}(Q-X_a))-cQ \end{aligned} \tag{A–4}$$

s.o.c.

$$\frac{\delta E[\Pi(H)]}{\delta Q]} = -H(F_{D_s}(Q-X_a)-1)-c \tag{A–5}$$

$$\frac{\delta^2 E[\Pi(H)]}{\delta Q^2]} = -Hf_{D_s}(Q-X_a) \leq 0 \tag{A–6}$$

f.o.c.

$$0 = -H(F_{D_s}(Q-X_a)-1)-c \tag{A–7}$$

$$F_{D_s}(Q-X_a) = \frac{H-c}{H} \tag{A–8}$$

$$Q^* = F_{D_s}^{-1}(\frac{H-c}{H})+X_a \tag{A–9}$$

Proof of Theorem 3: Optimal X_a^* for Bernoulli Customer Valuations with $p_s = L$ I show expected profit $E[\Pi(L)]$ to be convex in optimal advance sales inventory X_a^* as follows.

$$\begin{aligned} E[\Pi(L)] &= (L+(H-L)\alpha(\frac{M-Q}{M-X_a})-c_a)X_a \\ &\quad +L(Q-X_a)-cQ \end{aligned} \tag{A–10}$$

s.o.c.

$$\frac{\delta E[\Pi(L)]}{\delta X_a]} = \frac{(H-L)\alpha(M-Q)}{(M-X_a)}(1+\frac{X_a}{M-X_a})-c_a \tag{A–11}$$

$$\frac{\delta^2 E[\Pi(L)]}{\delta X_a^2]} = \frac{2(H-L)\alpha(M-Q)}{(M-X_a)^2}(1+\frac{X_a}{M-X_a}) \geq 0 \tag{A–12}$$

Assuming $H \geq L$ and $X_a \leq Q \leq M$, the s.o.c. is always positive, and thus the expected profit $E[\Pi(L)]$ is convex in X_a.

I then compare the extreme point solutions of $X_a = Q$ and $X_a = 0$ to derive the following condition.

$$\begin{aligned} E[\Pi(L, X_a = Q)] &\geq E[\Pi(L, X_a = 0)] \\ (L + (H-L)\alpha - c_a)Q - cQ &\geq Q(L - c) \\ ((H-L)\alpha - c_a)Q &\geq 0 \\ (H-L)\alpha &\geq c_a \end{aligned} \tag{A–13}$$

Proof of Theorem 4: Optimal X_a^* for Bernoulli Customer Valuations with $p_s = H$ I show an extreme point solution for X_a^*.

I determine that X_a^* must be an extreme point if the expected profit function is never increasing then decreasing.

Claim:

$$\begin{aligned} &\text{If } E[\Pi(Q^*, X_a + 1)] \geq E[\Pi(Q^*, X_a)] \\ &\text{then } E[\Pi(Q^*, X_a + 2)] \geq E[\Pi(Q^*, X_a + 1)],\ \forall X_a \end{aligned} \tag{A–14}$$

Where I define the following:

$$\begin{aligned} E[\Pi(H)] &= (L + (H-L)\alpha - c_a)X_a \\ &\quad + H((M - X_a)\alpha - \Lambda_{D_s}(Q - X_a)) - cQ \\ \Lambda_{D_s}(Q - X_a)) &= \int_{Q-X_a}^{\infty} (t - Q + X_a) f_{D_s}(t)dt \\ f_{D_s}(t) &= \frac{1}{\sigma_{D_s}\sqrt{2\pi}} e^{-(t-\mu_{D_s})^2/2\sigma_{D_s}^2} \\ \mu_{D_s} &= (M - X_a)\alpha \\ \sigma_{D_s} &= \sqrt{(M - X_a)\alpha(1-\alpha)} \end{aligned}$$

Since I know $Q^* = F^{-1}_{D_s(X_a)}\left(\frac{H-c}{H}\right) + X_a$, I can further define the following:

$$\Lambda_{D_s}(Q - X_a)) = \Lambda_{D_s}(F^{-1}_{D_s(X_a)}\left(\frac{H-c}{H}\right))) \tag{A–15}$$

$$F^{-1}_{D_s(X_a)}\left(\frac{H-c}{H}\right) = \mu_{D_s} + \sigma_{D_s} k \tag{A–16}$$

Where I define $k = \sqrt{2} erf^{-1}(2(\frac{H-c}{H}) - 1)$ as a constant.

Examining the conjecture of my claim, I have:

$$E[\Pi(Q^*, X_a + 1)] \geq E[\Pi(Q^*, X_a)] \tag{A–17}$$

$$\begin{aligned} H\Lambda_{D_s(X_a+1)}(F^{-1}_{D_s(X_a+1)}\left(\frac{H-c}{H}\right)) &\leq H\Lambda_{D_s(X_a)}(F^{-1}_{D_s(X_a)}\left(\frac{H-c}{H}\right)) \\ +ck\sigma_{X_a+1} - (L(1-\alpha) - c_a + c(1-\alpha)) & \quad +ck\sigma_{X_a} \end{aligned} \tag{A–18}$$

Since σ_{D_s} is always decreasing in X_a and it is reasonable to assume that $c_a \leq L + c$, this conjecture will be true when $\Lambda_{D_s(X_a+1)} \leq \Lambda_{D_s(X_a)}$. That is, expected profit is increasing when $\Lambda_{D_s(X_a)}$ is decreasing.

Thus, to prove my Claim, I show the following Lemma to be true.

Lemma:

$$\begin{aligned} &\text{If } \Lambda_{D_s(X_a+1)}(F^{-1}_{D_s(X_a+1)}\left(\frac{H-c}{H}\right))) \leq \Lambda_{D_s(X_a)}(F^{-1}_{D_s(X_a)}\left(\frac{H-c}{H}\right)) \\ &\text{then} \\ &\Lambda_{D_s(X_a+2)}(F^{-1}_{D_s(X_a+2)}\left(\frac{H-c}{H}\right))) \leq \Lambda_{D_s(X_a+1)}(F^{-1}_{D_s(X_a+1)}\left(\frac{H-c}{H}\right)) \end{aligned} \tag{A–19}$$

I derive the following conditions for $\Lambda_{D_s(X_a)}$ to be decreasing in X_a.

$$\frac{\delta}{\delta X_a}\Lambda_{D_s(X_a)}(F^{-1}_{D_s(X_a)}\left(\frac{H-c}{H}\right)) = \frac{\delta}{\delta X_a}\Lambda_{D_s(X_a)}(\mu_{D_s}+\sigma_{D_s}k) \leq 0 \tag{A–20}$$

if

$$\alpha(1-\alpha) \geq \frac{\sigma_{D_s}}{k-1/2} \tag{A–21}$$

and

$$\alpha \geq 1 - \frac{\mu_{D_s}(2\mu_{D_s}+3k\sigma_{D_s})-2}{k^2\mu_{D_s}} \tag{A–22}$$

and

$$\alpha \leq 1 - \frac{2\sigma^2_{D_s}}{\mu_{D_s}+k\sigma_{D_s}} \tag{A–23}$$

These conditions reduce to the following:

$$\begin{aligned} c &\leq 0.31H && \text{(A–24)}\\ \sigma_{D_s} &\leq \frac{1}{4}k-\frac{1}{8} && \text{(A–25)}\\ \alpha_{LB} &\leq \alpha \leq \alpha_{UB} && \text{(A–26)} \end{aligned}$$

where

$$\alpha_{LB} = \begin{cases} \frac{1-\sqrt{1-\frac{\sigma_{D_s}}{k-1/2}}}{2} & \text{, for } \frac{c}{H} \geq 0.00169 \\ 1-\frac{\mu_{D_s}(2\mu_{D_s}+3k\sigma_{D_s})-2}{k^2\mu_{D_s}} & \text{, otherwise} \end{cases} \tag{A–27}$$

$$\alpha_{UB} = \frac{1+\sqrt{1-\frac{\sigma_{D_s}}{k-1/2}}}{2} \tag{A–28}$$

Therefore, for α within the bounds defined above, I have $\Lambda_{D_s(X_a)}$ to be decreasing in X_a, and thus expected profit to be increasing in X_a. From my Claim, if the expected profit is increasing (that is α meets the conditions defined above, then it is always increasing. Thus, I have an extremem point solution for X_a^*.

I then compare the extreme point values of X_a to determine which policy will yield the largest expected profit.

$$E[\Pi(H)] = (L + (H - L)\alpha - c_a)X_a + H((M - X_a)\alpha - \Lambda_{D_s}(Q - X_a)) - cQ \quad \text{(A–29)}$$

$$E[\Pi(H, X_a = Q)] = (L + (H - L)\alpha - c_a)Q + H((M - Q)\alpha - \Lambda_{D_s}(0)) - cQ \quad \text{(A–30)}$$

$$E[\Pi(H, X_a = 0)] = H((M)\alpha - \Lambda_{D_s}(Q)) - cQ \quad \text{(A–31)}$$

(A–32)

$$E[\Pi(H, X_a = Q)] \geq E[\Pi(H, X_a = 0)] \quad \text{(A–33)}$$

$$Q[L(1 - \alpha) - c_a] \geq H[\mu_{D_s} - \Lambda_{D_s}(Q)] \quad \text{(A–34)}$$

Since $\Lambda_{D_s(0)}(0) = \int_0^\infty t f_{D_s}(t)dt = \mu_{D_s}$.

Proof of Theorem 5: Optimal Policy (Q^*, X_a^*) for Bernoulli Customer Valuations with $p_s = L$ I derive the optimal order policy (Q^*, X_a^*) for $p_s = L$ and $Q^*(L) = M$ as follows.

$$E[\Pi(Q^*(L) = M)] = (L - c_a)X_a + L(M - X_a) - cM \quad \text{(A–35)}$$

s.o.c.

$$\frac{\delta E[\Pi(Q^*(L) = M)]}{\delta X_a]} = -c_a \quad \text{(A–36)}$$

$$\frac{\delta^2 E[\Pi(Q^*(L) = M)]}{\delta X_a^2]} = 0 \quad \text{(A–37)}$$

Since the f.o.c. is negative, expected profit is decreasing in X_a; thus, $X^*(L) = 0$. I compare the expected profit for the $(M, 0)$ and $(0, 0)$ policies to determine the optimal policy. I clearly find the $(M, 0)$ policy to be optimal.

$$E[\Pi(L, Q = M, X_a = 0)] = (L - c)M \quad \text{(A–38)}$$

$$E[\Pi(L, Q = 0, X_a = 0)] = 0 \quad \text{(A–39)}$$

$$\quad \text{(A–40)}$$

Proof of Theorem 6: Optimal Policy (Q^*, X_a^*) **for** $p_s = H$ I find an extreme point solution for X_a as done in Proof A.

I find the upper bound of $X_a(H)$ as follows, assuming $X_a \leq Q$ and the maximum Q value is M.

$$\begin{aligned} Q &= F_{D_s}^{-1}(\frac{H-c}{H}) + X_a \\ &= \mu_{D_s} + \sigma_{D_s} k + X_a \end{aligned} \quad \text{(A–41)}$$

Where I define $k = \sqrt{2} erf^{-1}(2(\frac{H-c}{H}) - 1)$ as a constant, independent of X_a. And I have $\mu_{D_s} = (M - X_a)\alpha$ and $\sigma_{D_s} = \sqrt{(M - X_a)\alpha(1-\alpha)}$.

If $Q = M$, its maximum value, then the maximum values of X_a is found as follows.

$$\begin{aligned} M &= \mu_{D_s} + \sigma_{D_s} k + X_a \\ &= (M - X_a)\alpha + \sqrt{(M - X_a)\alpha(1-\alpha)}k + X_a \\ X_a &= M - \frac{\alpha k^2}{1-\alpha} \end{aligned} \quad \text{(A–42)}$$

If X_a is at its other extreme, 0, then $Q^* = F_{D_s}^{-1}\left(\frac{H-c}{H}\right)$.

I now compare the $(M, M-\frac{\alpha k^2}{1-\alpha})$ and $(F_{D_s}^{-1}\left(\frac{H-c}{H}\right), 0)$ policies as follows.

$$\begin{aligned} E[\Pi(Q=M, X_a = M-\frac{\alpha k^2}{1-\alpha})] &= (L+(H-L)\alpha - c_a)(M-\frac{\alpha k^2}{1-\alpha}) \\ &\quad + H(\frac{\alpha^2 k^2}{1-\alpha} - \Lambda_{D_s}(F_{D_s}^{-1}\left(\frac{H-c}{H}\right))) \\ &\quad -cM \qquad \text{(A–43)} \\ E[\Pi(Q=F_{D_s}^{-1}\left(\frac{H-c}{H}\right), X_a = 0)] &= H(M\alpha - \Lambda_{D_s}(F_{D_s}^{-1}\left(\frac{H-c}{H}\right))) \\ &\quad -cF_{D_s}^{-1}\left(\frac{H-c}{H}\right) \qquad \text{(A–44)} \end{aligned}$$

$$\begin{aligned} E[\Pi(Q=M, X_a = M-\frac{\alpha k^2}{1-\alpha})] &\geq E[\Pi(Q=F_{D_s}^{-1}\left(\frac{H-c}{H}\right), X_a=0)] \qquad \text{(A–45)} \\ L(1-\alpha)+c &\geq c_a \qquad \text{(A–46)} \end{aligned}$$

Proof of Theorem 7: Optimal Spot Price p_s^* Bernoulli Customer Valuations I compare the expected profit under the optimal order policies for $p_s = L$ and $p_s = H$.

$$\begin{aligned} E[\Pi(L)] &= (L+(H-L)\alpha(\frac{M-Q}{M-X_a}) - c_a)X_a + L(Q-X_a) - cQ \\ (Q^*(L), X_a^*(L)) &= (M, 0) \\ E[\Pi(Q^*(L), X_a^*(L))] &= (L+(H-L)\alpha(\frac{M-M}{M-0}) - c_a)0 + L(M-0) - cM \\ &= (L-c)M \qquad \text{(A–47)} \end{aligned}$$

$$\begin{aligned} E[\Pi(H)] &= (L+(H-L)\alpha - c_a)X_a + H((M-X_a)\alpha - \Lambda_{D_s}(Q-X_a)) - cQ \\ (Q^*(H), X_a^*(H)) &= (F_{D_s}^{-1}(\frac{H-c}{H}) + X_a^*(H), X_a^*(H)) \\ E[\Pi(Q^*(H), X_a^*(H))] &= (L+(H-L)\alpha - c_a)X_a^*(H) + H((M-X_a^*(H))\alpha - \Lambda_{D_s}(F_{D_s}^{-1}(\frac{H-c}{H}))) \\ &\quad -c(F_{D_s}^{-1}(\frac{H-c}{H}) + X_a^*(H)) \qquad \text{(A–48)} \end{aligned}$$

Proof of Theorem 8: Optimal Order Quantity Q^* for Uniform Customer Valuations I show expected profit to be concave in Q for Uniform customer valuations.

$$\begin{aligned} E[\Pi] &= (\frac{H+L}{2} - \frac{(H-p_s)^2}{2(H-L)} + \frac{\Lambda_{D_s}(Q-X_a)(H-p_s)}{2(M-X_a)} - c_a)X_a \\ &\quad + p_s((M-X_a)\frac{H-p_s}{H-L} - \Lambda_{D_s}(Q-X_a)) - cQ \qquad \text{(A–49)} \\ \frac{\delta E[\Pi]}{\delta Q} &= \left(\frac{Xa(H-p_s)}{2(M-X_a)} - p_s\right)[F_{D_s}(Q-X_a)-1] - c \qquad \text{(A–50)} \\ \frac{\delta^2 E[\Pi]}{\delta Q^2} &= \left(\frac{Xa(H-p_s)}{2(M-X_a)} - p_s\right) f_{D_s}(Q-Xa) \leq 0 \qquad \text{(A–51)} \end{aligned}$$

I can observe that the s.o.c. is concave for $\left(\frac{Xa(H-p_s)}{2(M-X_a)} - p_s\right) \leq 0$. Assuming this condition to hold, I solve the f.o.c. to find the optimal Q^* value as follows.

$$Q^* = F_{D_s}^{-1}(\frac{p_s - c - \frac{X_a(H-p_s)}{2(M-X_a)}}{p_s - \frac{X_a(H-p_s)}{2(M-X_a)}}) + X_a \qquad \text{(A–52)}$$

APPENDIX B
PROOFS FOR CHAPTER 4: MULTI-GENERATION PRICING AND TIMING DECISIONS IN NEW PRODUCT DEVELOPMENT

Proof of Theorem 1: Given the sales rate expressions for CASE 1, I derive the following:

$$\frac{\delta f}{\delta p_1} = -a_1 \tag{B–1}$$

$$\frac{\delta g}{\delta p_2} = -b_1 \tag{B–2}$$

Since I have $\frac{\delta f}{\delta x_1} = \frac{\delta g}{\delta x_2} = 0$, the following solutions for marginal sales λ_1 and λ_2 directly follow.

$$\lambda_1 = 0, \forall t \tag{B–3}$$

$$\lambda_2 = 0, \forall t \tag{B–4}$$

I can then derive the optimal prices as follows:

$$p_1^* : \quad f(x_1, p_1) + (p_1 - c_1 + \lambda_1)\frac{\delta f}{\delta p_1} = 0 \tag{B–5}$$

$$a_0 - a_1 p_1 - a_1(p_1 - c_1) = 0 \tag{B–6}$$

$$p_1^* = \frac{1}{2}(\frac{a_0}{a_1} + c_1) \tag{B–7}$$

$$p_2^* : \quad g(x_2, p_2) + (p_2 - c_2 + \lambda_2)\frac{\delta g}{\delta p_2} = 0 \tag{B–8}$$

$$b_0 - b_1 p_2 - b_1(p_2 - c_2) = 0 \tag{B–9}$$

$$p_2^* = \frac{1}{2}(\frac{b_0}{b_1} + c_2) \tag{B–10}$$

Proof of Theorem 2: To determine the optimal time to market, t_m^*, I derive $\lambda_3(t_m)$ as follows.

$$\dot{\lambda}_3 = (p_1 - c_1 + \lambda_1)f(x_1, p_1) - (p_2 - c_2 + \lambda_2)g(x_2, p_2) \tag{B–11}$$

$$= (p_1 - c_1)(a_0 - a_1 p_1) - (p_2 - c_2)(b_0 - b_1 p_2) \tag{B–12}$$

Let us define $K_1 = (p_1 - c_1)(a_0 - a_1 p_1)$ and $K_2 = (p_2 - c_2)(b_0 - b_1 p_2)$ as the profit rate for each generation. I can then solve for $\lambda_3(t_m)$ using backwards integration (given $\lambda_3(T) = 0$).

$$\lambda_3(t_m) = \lambda_3(T) - \int_{t_m}^{T} \dot{\lambda}_3 dt \tag{B–13}$$

$$= -\int_{t_m}^{T} (K_1 - K_2) dt \tag{B–14}$$

I can then determine the optimal value of t_m^* from the condition $\lambda_3(t_m) \geq c_{t_m}$.

Proof of Theorem 3: Given the sales rate expressions for CASE 2, I can derive the following:

$$\frac{\delta f}{\delta x_1} = a_2[\psi - \phi - 2\frac{\psi}{M_1}x_1] \tag{B–15}$$

$$\frac{\delta g}{\delta x_2} = b_2[\psi - \phi - 2\frac{\psi}{M_2}x_2] \tag{B–16}$$

Since I have $\frac{\delta f}{\delta p_1} = \frac{\delta g}{\delta p_2} = 0$, the optimal prices immediately follow.

$$p_1^* = \begin{cases} \hat{p}_1, & f(x_1, p_1) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{B–17}$$

$$p_2^* = \begin{cases} \hat{p}_2, & g(x_2, p_2) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{B–18}$$

Where $\hat{p}_1$ and $\hat{p}_2$ be defined as some maximum price levels set by the market. Since I define $f(x_1, p_1)$ and $g(x_2, p_2)$ as positive functions, the above conditions will always hold.

Proof of Theorem 4: To solve for the optimal time to market, t_m^*, I derive the following differential equation solutions for the marginal value of sales of each generation and the cumulative sales of each generation:

$$\lambda_1 = \begin{cases} (p_1 - c_1)\left(\frac{e^{-a_2(\phi-\psi)(t-t_m)}(e^{a_2(\phi+\psi)t}\phi+\psi)^2}{(e^{a_2(\phi+\psi)t_m}\phi+\psi)^2} - 1\right), & t < t_m, \\ \lambda_1(t) = \lambda_1(T) = 0, & t \geq t_m. \end{cases} \tag{B–19}$$

$$\lambda_2 = \begin{cases} \lambda_2(0) = (p_2(t_m) - c_2)\left(\frac{e^{-b_2(\phi+\psi)T}(\phi+\psi)^2}{(e^{b_2(\phi+\psi)T}\phi+\psi)^2} - 1\right), & t < t_m, \\ (p_2 - c_2)\left(\frac{e^{-b_2(\phi+\psi)(t-T)}(e^{b_2(\phi+\psi)t}\phi+e^{b_2(\phi+\psi)t_m}\psi)^2}{(e^{b_2(\phi+\psi)T}\phi+e^{b_2(\phi+\psi)t_m}\psi)^2} - 1\right), & t_m \le t < T, \\ \lambda_2(T) = 0, & t = T. \end{cases} \tag{B–20}$$

$$x_1 = \begin{cases} \frac{M_1(1-e^{-a_2 t(\phi+\psi)})}{1+\frac{\psi}{\phi}e^{-a_2 t(\phi+\psi)}}, & t < t_m, \\ x_1(t_m) = \frac{M_1(1-e^{-a_2 t_m(\phi+\psi)})}{1+\frac{\psi}{\phi}e^{-a_2 t_m(\phi+\psi)}}, & t \ge t_m. \end{cases} \tag{B–21}$$

$$x_2 = \begin{cases} x_2(0) = 0, & t < t_m, \\ \frac{M_2(1-e^{-b_2(t-t_m)(\phi+\psi)})}{1+\frac{\psi}{\phi}e^{-b_2(t-t_m)(\phi+\psi)}}, & t \ge t_m. \end{cases} \tag{B–22}$$

(Note that the expressions for x_1 and x_2 are similar to those found in the literature. Refer to Krishnan, Bass, and Jain (1999) [14].)

I then derive $\lambda_3(t_m)$ using backwards integration. Recall that $\lambda_3(t_m)$ represents the marginal benefit of introducing Generation 2 and earning Generation 2 sales for the remaining time versus continuing to sell Generation 1.

$$\lambda_3(t_m) = \lambda_3(T) - \int_{t_m}^{T} \dot{\lambda}_3(\tau)d\tau \tag{B–23}$$

Where $\lambda_3(T) = 0$. Using the above solutions for $\lambda_1, \lambda_2, x_1$, and x_2, I can derive $\dot{\lambda}_3$ as follows:

$$\dot{\lambda}_3 = (p_1 - c_1 + \lambda_1)f(x_1, p_1) - (p_2 - c_2 + \lambda_2)g(x_2, p_2) \tag{B–24}$$

$$= \begin{cases} (p_1 - c_1)f(x_1(t_m)) - (p_2 - c_2 + \lambda_2(t_m))(b_2\phi M_2), & t = t_m, \\ (p_1 - c_1)f(x_1(t_m)) - (p_2 - c_2 + \lambda_2(t))g(x_2(t)), & t_m < t \le T. \end{cases} \tag{B–25}$$

I can then write $\lambda_3(t_m)$ as follows.

$$\begin{aligned}\lambda_3(t_m) &= \phi(\phi+\psi)^2[-\frac{a_2 M_1 e^{a_2(\phi+\psi)t_m}(p_1-c_1)}{(e^{a_2(\phi+\psi)t_m}\phi+\psi)^2} \\ &+\frac{b_2 M_2 e^{b_2(\phi+\psi)(T+t_m)}(p_2-c_2)}{(e^{b_2(\phi+\psi)T}\phi+e^{b_2(\phi+\psi)t_m}\psi)^2}](T-t_m) \qquad \text{(B–26)}\end{aligned}$$

This can be simplified to the following expression:

$$\lambda_3(t_m) = [(p_2-c_2)g(x_2(T-t_m))-(p_1-c_1)f(x_1(t_m))](T-t_m) \qquad \text{(B–27)}$$

I then find t_m^* from the condition that $\lambda_3(t_m) \geq c_{t_m}$.

Proof of Theorem 5: I solve for this threshold value by determining the smallest value of T for which $\lambda_3(t_m=0) \geq c_{t_m}$.

$$\lambda_3(t_m=0) = \phi(\phi+\psi)^2(-\frac{(a_2M_1(p_1-c_1))}{(\phi+\psi)^2}+\frac{(b_2 e^{b_2(\phi+\psi)T}M_2(p_2-c_2))}{(e^{b_2(\phi+\psi)T}\phi+\psi)^2})T \geq c_{t_m} \qquad \text{(B–28)}$$

When the cost of bringing the second generation to market is negligible ($c_{t_m}=0$), I can derive a value for $\bar{T}$.

Proof of Theorem 6: The following relationships are derived from the first order conditions of optimality:

$$\frac{\delta f}{\delta x_1} = a_2[\psi-\phi-2\frac{\psi}{M_1}x_1] \qquad \text{(B–29)}$$

$$\frac{\delta f}{\delta p_1} = -a_1 \qquad \text{(B–30)}$$

$$\frac{\delta g}{\delta x_2} = b_2[\psi-\phi-2\frac{\psi}{M_2}x_2] \qquad \text{(B–31)}$$

$$\frac{\delta g}{\delta p_2} = -b_1 \qquad \text{(B–32)}$$

I begin by examining the optimal price expression to find $p_1^*(x_1, \lambda_1)$.

$$\begin{aligned} p_1^*: \qquad & f(x_1, p_1) + (p_1 - c_1 + \lambda_1)\frac{\delta f}{\delta p_1} = 0 \\ & a_0 - a_1 p + a_2[\phi(M_1 - x_1) + \frac{\psi}{M_1}(M_1 - x_1)x_1] - a_1(p_1 - c_1 + \lambda_1) = 0 \end{aligned} \tag{B–33}$$

$$p_1^*(x_1, \lambda_1) \;=\; -\left(\frac{-M_1(a_0 + a_1 c_1 + a_2 M_1 \phi) + a_1 M_1 \lambda_1 + a_2 x_1(M_1(\phi - \psi) + \psi x_1)}{2a_1 M_1}\right) \tag{B–34}$$

I replace p_1 with $p_1^*(x_1, \lambda_1)$ in the expressions for $\dot{x}_1$ and $\dot{\lambda}_1$. Now I solve simultaneous differential equations to find x_1 and λ_1.

$$\begin{aligned} \dot{x}_1 &= f(x_1, p_1^*(x_1, \lambda_1)) \\ &= a_0 - a_1 p_1^*(x_1, \lambda_1) + a_2[\phi(M_1 - x_1) + \frac{\psi}{M_1}(M_1 - x_1)x_1] \end{aligned} \tag{B–35}$$

$$\begin{aligned} \dot{\lambda}_1 &= -(p_1^*(x_1, \lambda_1) - c_1 + \lambda_1)\frac{\delta f}{\delta x_1} \\ \dot{\lambda}_1 &= -(p_1^*(x_1, \lambda_1) - c_1 + \lambda_1)a_2[\psi - \phi - 2\frac{\psi}{M_1}x_1] \end{aligned} \tag{B–36}$$

$$x_1 = \frac{\left(\begin{array}{l} t[a_1 M_1(-2 + a_2(-\phi + \psi)t_m) \\ \pm\sqrt{a_1^2 M_1(4a_2(a_0 - a_1 c_1)\psi t_m^2 + M_1(4 + a_2 t_m(4\phi - 4\psi + a_2(\phi + \psi)^2 t_m)))}] \end{array}\right)}{2a_1 a_2 \psi t_m^2} \tag{B–37}$$

$$\lambda_1 = \frac{\left(\begin{array}{l} (-t + t_m)[2a_1^2 a_2 c_1 \psi t_m^2(t + t_m) \\ +(2t_m + t(2 + a_2(\phi - \psi)t_m)) \\ \sqrt{a_1^2 M_1(4a_2(a_0 - a_1 c_1)\psi t_m^2 + M_1(4 + a_2 t_m(4\phi - 4\psi + a_2(\phi + \psi)^2 t_m)))} \\ \pm a_1(2a_0 a_2 \psi t_m^2(t + t_m) + M_1(2t_m(2 + a_2 t_m(\phi - \psi + a_2 \phi\psi t_m)) \\ +t(4 + a_2 t_m(4\phi - 4\psi + a_2(\phi^2 + \psi^2)t_m))))] \end{array}\right)}{2a_1^2 a_2 \psi t_m^4} \tag{B–38}$$

I can now replace these solutions for x_1 and λ_1 in my expression for $p_1^*(x_1, \lambda_1)$ to find p_1^*.

Likewise, I can repeat this analysis to find x_2, λ_2, and p_2^*.

$$p_2^* : \qquad g(x_2, p_2) + (p_2 - c_2 + \lambda_2)\frac{\delta g}{\delta p_2} = 0$$

$$b_0 - b_1 p_2 + b_2[\phi(M_2 - x_2) + \frac{\psi}{M_2}(M_2 - x_2)x_2] - b_1(p_2 - c_2 + \lambda_2) = 0 \tag{B–39}$$

$$p_2^*(x_2, \lambda_2) = -\left(\frac{-M_2(b_0 + b_1 c_2 + b_2 m_2 \phi) + b_1 M_2 \lambda_2 + b_2 x_2 (M_2(\phi - \psi) + \psi x_2)}{2 b_1 M_2}\right) \tag{B–40}$$

$$\begin{aligned} \dot{x}_2 &= g(x_2, p_2^*(x_2, \lambda_2)) \\ &= b_0 - b_1 p_2^*(x_2, \lambda_2) + b_2[\phi(M_2 - x_2) + \frac{\psi}{M_2}(M_2 - x_2)x_2] \end{aligned} \tag{B–41}$$

$$\begin{aligned} \dot{\lambda}_2 &= -(p_2^*(x_2, \lambda_2) - c_2 + \lambda_2)\frac{\delta g}{\delta x_2} \\ &= -(p_2^*(x_2, \lambda_2) - c_2 + \lambda_2) b_2[\psi - \phi - 2\frac{\psi}{M_2}x_2] \end{aligned} \tag{B–42}$$

$$x_2 = \frac{\left(\begin{array}{l} (t-t_m)[b_1 M_2(-2-b_2(\phi-\psi)(T-t_m)) \\ \pm \sqrt{\begin{array}{l} b_1^2 M_2(M_2(4+4b_2(\phi-\psi)(T-t_m)+b_2^2(\phi+\psi)^2(T-t_m)^2) \\ +4b_2(b_0-b_1c_2)\psi(T-t_m)^2) \end{array}}\ ] \end{array} \right)}{2b_1b_2\psi(T-t_m)^2} \quad \text{(B–43)}$$

$$\lambda_2 = \frac{\left(\begin{array}{l} (T-t)[2b_1^2 b_2 c_2\psi(t+T-2t_m)(T-t_m)^2 \\ \pm \sqrt{\begin{array}{l} b_1^2 M_2(M_2(4+4b_2(\phi-\psi)(T-t_m)+b_2^2(\phi+\psi)^2(T-t_m)^2) \\ +4b_2(b_0-b_1c_2)\psi(T-t_m)^2) \end{array}} \\ (t(2+b_2(\phi-\psi)(T-t_m))+2(T-2t_m)-b_2(\phi-\psi)(T-t_m)t_m) \\ -b_1(2b_0b_2\psi(t+T-2t_m)(T-t_m)^2+m_2(t(4+4b_2(\phi-\psi)(T-t_m) \\ +b_2^2(\phi^2+\psi^2)(T-t_m)^2)+4(T-2t_m)-b_2(T-t_m)(-2T(\phi-\psi+b_2\phi\psi T) \\ +(6\phi-6\psi+b_2(\phi^2+4\phi\psi+\psi^2)T)t_m-b_2(\phi+\psi)^2t_m^2)))] \end{array} \right)}{2b_1^2b_2\psi(T-t_m)^4}$$

(B–44)

Proof of Theorem 7: I derive the expression for $\lambda_3(t_m)$ as follows.

$$\dot{\lambda}_3(p_1^*, p_2^*) = (p_1^* - c_1 + \lambda_1)f(x_1, p_1^*) - (p_2^* - c_2 + \lambda_2)g(x_2, p_2^*)$$

(B–45)

I know that $\lambda_3(T) = 0$ and for $t \geq t_m$, I have $\lambda_1(t) = 0$, $x_1(t) = x_1(t_m)$, and $p_1(t) = p_1(t_m)$. I now solve for $\lambda_3(t_m)$ as follows:

$$\begin{aligned} &\lambda_3(T) - \int_{t_m}^{T} \dot{\lambda}_3(\tau)d\tau \\ &\qquad = -\int_{t_m}^{T} [(p_1^*(t_m) - c_1)f(x_1(t_m), p_1^*(t_m)) \\ &\qquad\quad -(p_2^*(\tau) - c_2 + \lambda_2(\tau))g(x_2(\tau), p_2^*(\tau))]d\tau \end{aligned}$$

APPENDIX C
PROOFS FOR CHAPTER 5: OPTIMAL NUMBER OF GENERATIONS FOR A MULTI-GENERATION PRICING AND TIMING MODEL IN NEW PRODUCT DEVELOPMENT

Proof of Theorem 1: Optimal Price To find the optimal price $p^*(t)$, we first use Optimal Control to derive an expression for p^* in terms of cumulative sales x and the marginal value of sales λ.

$$\frac{\delta H}{\delta p} = -a_1(\lambda + n(p-c)) + n(a_0 - a_1 p + a_2(\phi(M-x) + \frac{\psi}{M}(M-x)x)) \tag{C–1}$$

$$p^* = \frac{Mn(a_0 + a_2 M\phi) + a_1 M(nc - \lambda) - a_2 nx(M(\phi - \psi) + \psi x)}{2a_1 Mn} \tag{C–2}$$

We then solve for x and λ using simultaneous differential equations and Optimal Control methods.

$$\frac{\delta H}{\delta x} = (n(p-c) + \lambda)(M(\phi - \psi) + 2\psi x)\frac{a_2}{M} \tag{C–3}$$

$$x^*(t) = \frac{t\left(a_1 Mn(-2 + a_2(\psi - \phi)t_m) + \sqrt{\begin{array}{l} a_1^2 Mn^2(4a_2(a_0 - a_1 c)\psi t_m^2 \\ +M(4 + a_2 t_m(4\phi - 4\psi + a_2(\phi + \psi)^2 t_m))) \end{array}}\right)}{2a_1 a_2 n\psi t_m^2} \tag{C–4}$$

$$\lambda^*(t) = \frac{\left(\begin{array}{l} (t_m - t)(2a_1^2 a_2 cn\psi t_m^2(t + t_m) + (2t_m + t(2 + a_2(\phi - \psi)t_m)) \\ \sqrt{a_1^2 Mn^2(4a_2(a_0 - a_1 c)\psi t_m^2 + M(4 + a_2 t_m(4\phi - 4\psi + a2(\phi + \psi)^2 t_m)))} \\ -a_1 n(2a_0 a_2 \psi t_m^2(t + t_m) + M(2t_m(2 + a_2 t_m(\phi - \psi + a_2\phi\psi t_m)) \\ +t(4 + a_2 t_m(4\phi - 4\psi + a_2(\phi^2 + \psi^2)t_m))))) \end{array}\right)}{2a_1^2 a_2 \psi t_m^4} \tag{C–5}$$

We then substitute back in the derived x^* and λ^* expressions to find the expression for p^*.

Proof of Corollary 1: Optimal Price Concave over Time The second derivative of the optimal price $p^*(t)$ over time is the following:

$$\frac{\delta^2 p^*(t)}{\delta t^2} = \frac{\begin{array}{l}(2a_1^2 a_2 cn\psi t_m^2 + (2 + a_2(\phi - \psi)t_m) \\ \sqrt{(a_1^2 M n^2(4a_2(a_0 - a_1 c)\psi t_m^2 + M(4 + a_2 t_m(4\phi - 4\psi + a_2(\phi + \psi)^2 t_m))))} \\ -a_1 n(2a_0 a_2 \psi t_m^2 + M(4 + a_2 t_m(4\phi - 4\psi + a_2(\phi^2 + \psi^2)t_m))))\end{array}}{a_1^2 a_2 n \psi t_m^4} \tag{C–6}$$

We want to show that $\frac{\delta^2 p^*(t)}{\delta t^2} \leq 0$ in order to meet the s.o.c. for concavity. Let us define the following:

$$Y = 2a_1 a_2 n \psi t_m^2 (a_1 c - a_0) \tag{C–7}$$

$$J = 2 + a_2 t_m(\phi - \psi) \tag{C–8}$$

$$\alpha = a_1 n M \tag{C–9}$$

$$\beta = 2a_2^2 t_m^2 \phi \psi \tag{C–10}$$

Then we can rewrite the s.o.c. as follows:

$$\frac{Y + J\sqrt{\Delta} - \alpha(J^2 + \beta)}{a_1^2 a_2 n \psi t_m^4} \leq 0 \tag{C–11}$$

Where $\Delta = \alpha(\alpha(J^2 + 2\beta) - 2Y)$.

We can rearrange the s.o.c. expression as follows:

$$J\sqrt{\Delta} \leq \alpha(J^2 + \beta) - Y \tag{C–12}$$

If we assume $a_0 \geq a_1 c$, we have $Y \leq 0$. This assumption is reasonable for typical values of a_1, a_2, and c based on the empirical literature. If $Y \leq 0$, then $-Y \geq 0$ and the RHS of expression C.12 is positive. Now we must note that it is possible for J to be negative. From the empirical literature, typical values of ϕ are much smaller than typical values of ψ. Thus, we usually have $\phi << \psi$ which implies that $(\phi - \psi)$ would be negative. Thus, if

$t_m \leq \frac{2}{a_2(\psi - \phi)}$, then $J \leq 0$. If this is the case, and J is negative, then the LHS of expression C.12 is negative and the s.o.c. for concavity of $p(t)$ in t always holds. However, if the condition $t_m \leq \frac{2}{a_2(\psi - \phi)}$ is not true and J is positive, then we can perform the following analysis:

$$J^2\Delta \leq (\alpha(J^2 + \beta) - Y)^2 \quad \text{(C–13)}$$

$$J^2(\alpha^2 J^2 + \alpha^2 2\beta - \alpha 2Y) \leq \alpha^2(J^4 + 2J^2\beta + \beta^2) - 2Y\alpha(J^2 + \beta) + Y^2 \quad \text{(C–14)}$$

$$0 \leq (\alpha\beta - Y)^2 \quad \text{(C–15)}$$

Since the expression C.15 is always true, the s.o.c. for concavity of $p(t)$ in t always holds. Thus, regardless of whether J is positive or negative, the s.o.c. always holds.

We must also ensure that the square root expression is real by checking that $\Delta \geq 0$. This requires the following condition to be true.

$$\begin{aligned} \Delta &\geq 0 \\ \alpha(J^2 + 2\beta) &\geq 2Y \end{aligned} \quad \text{(C–16)}$$

Since we have $Y \leq 0$ when $a_0 - a_1 c \geq 0$, then the expression in C.16 is always true and therefore Δ is always positive.

Thus, for $a_0 - a_1 c \geq 0$, the s.o.c. is true and $p^*(t)$ is concave in t.

Proof of Theorem 2: Optimal Number of Generations Substituting the optimal price expression found in Theorem 1 into our profit expression yields the following:

$$\Pi(p^*(t)) = n\int_0^{T/n} \dot{x}(p^*(t))(p^*(t)-c(t))dt - nc_{t_m} \tag{C–17}$$

$$= \frac{\left(\begin{array}{l} (a_1Mn(-2+a_2(\psi-\phi)t_m) \\ +\sqrt{\begin{array}{l} a_1^2Mn^2(4a_2(a_0-a_1c)\psi t_m^2 \\ +M(4+a_2t_m(4\phi-4\psi+a_2(\phi+\psi)^2t_m))) \end{array}}) \\ (-((cT)/n)+1/(12a_1^2a_2n^4\psi t_m^4)(T(4a_1^2a_2cn\psi T^2t_m^2 \\ +(-6n^2t_m^2+3a_2n(\psi-\phi)Tt_m^2+2T^2(2+a_2(\phi-\psi)t_m)) \\ \sqrt{\begin{array}{l} a_1^2Mn^2(4a_2(a_0-a_1c)\psi t_m^2 \\ +M(4+a_2t_m(4\phi-4\psi+a_2(\phi+\psi)^2t_m))) \end{array}} \\ +a_1n(4a_0a_2\psi t_m^2(-T^2+3n^2t_m^2) \\ +M(3a_2n(\phi-\psi)Tt_m^2(2+a_2(\phi-\psi)t_m) \\ +6n^2t_m^2(2+a_2t_m(\phi-\psi+2a_2\phi\psi t_m)) \\ -2T^2(4+a_2t_m(4\phi-4\psi+a_2(\phi^2+\psi^2)t_m))))))) \end{array}\right)}{2a_1a_2\psi t_s m^2} - nc_{t_m} \tag{C–18}$$

Assuming a benchmark scenario of equal generations offered for equal time to market for each generation, we replace t_m with T/n. Making this substitution in Π, we can find n^* using the f.o.c.: $\frac{\delta\Pi}{\delta n} = 0$.

$$\frac{\delta\Pi}{\delta n} = \frac{\begin{array}{l} (16Mn^2+10a_2Mn(\phi-\psi)T+a_2(4a_0\psi+a_2M(\phi+\psi)^2)T^2-4a_1a_2c\psi T^2) \\ \sqrt{a_1^2M(4Mn^2+4a_2Mn(\phi-\psi)T+a_2(4(a_0-a_1c)\psi+a_2M(\phi+\psi)^2)T^2)} \\ +a_1M(-32Mn^3-36a_2Mn^2(\phi-\psi)T-12a_2n(2a_0\psi+a_2M(\phi^2+\psi^2))T^2 \\ -a_2^2(\phi-\psi)(6a_0\psi+a_2M(\phi^2+4\phi\psi+\psi^2))T^3) \\ +6a_1^2a_2\psi T^2(-2a_2c_{t_m}\psi T+cM(4n+a_2(\phi-\psi)T))) \end{array}}{12a_1^2a_2^2\psi^2T^3} \tag{C–19}$$

The optimal expression for n^* is derived by setting the above expression to zero and solving for n.

Second Order Condition for Profit to be Concave in n In order to have n^* from Theorem 2, derived from the f.o.c. of profit in n, to be optimal, we must first show that the s.o.c. of profit in n holds. That is, we must have $\frac{\delta^2 \Pi}{\delta n^2} \leq 0$ true in order for profit to be concave in n and thus n^* to be the n value which maximizes the profit function. The expression for this s.o.c. is as follows:

$$\begin{aligned}
&\frac{\delta^2 \Pi}{\delta n^2} \leq 0 \\
&\left(\begin{array}{l} -8Mn^2(K-2Mn) + 2a_2Mn(-2K+10Mn(\phi-\psi)T \\ -a_2(2(a_0-a_1c)(K-6Mn)\psi + a_2M(4Mn(-2\phi^2+\phi\psi-2\psi^2) \\ +K(\phi^2+\psi^2)))T^2 + a_2^2M(\phi-\psi)(4(a_0-a_1c)\psi + a_2M(\phi-\psi)^2)T^3 \end{array} \right) \leq 0
\end{aligned} \tag{C–20}$$

Where we define

$$K = \sqrt{M(4Mn^2 + 4a_2Mn(\phi-\psi)T + a_2(4(a_0-a_1c)\psi + a_2M(\phi+\psi)^2)T^2)}$$

Although we cannot show analytically that this expression is always true, nor derive analytical conditions under which this expression would hold, given numerical values for the expression parameters, this condition can easily be checked. If this condition holds, then n^* found in Theorem 2 is indeed the optimal number of generations to offer.

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BIOGRAPHICAL SKETCH

Michelle M.H. Şeref received her Bachelor of Science in industrial and systems engineering in 2002, and her Master of Science in industrial and systems engineering in 2004, both from the University of Florida. Upon completion of her Ph.D. in Operations Management from the University of Florida, Michelle plans to pursue an academic career.

Michelle has published various works in the Operations Management and Industrial Engineering literature. In the Operations Management area, she has completed a paper titled "Option Requirements Planning for Mass Customized Products" with Drs. Anand Paul and Asoo Vakharia, and is working on a review paper of extensions to the newsvendor problem with Dr. Vakharia and several other colleagues. In the Industrial Engineering area, Michelle has published the journal article "Decision support systems development: An essential part of OR education" with Dr. Ravindra Ahuja, in addition to a chapter titled "Spreadsheet-Based Decision Support Systems" published in the book *Handbook on Decision Support Systems*. Michelle has also co-authored a textbook *Developing Spreadsheet-Based Decision Support Systems (DSS) Using Excel and VBA for Excel* with Dr. Ahuja and Dr. Wayne Winston. Michelle plans to submit the three research chapters of this dissertation as publications in the Operations Management (OM) literature. Her research focuses on joint operations and pricing decisions in the OM/Marketing interface area.

Michelle has taught several courses in both the Industrial Engineering and Information Systems and Operations Management Departments at the University of Florida. These courses include Project Management course, Managerial Operations Analysis, and Developing Decision Support Systems for both undergraduate and graduate students. She has received high evaluations in all of her teaching.

Michelle enjoys examining advanced-level research problems, teaching courses to students of all levels, and interacting with colleagues at national and international conferences. She looks forward to an exciting academic career.